

# Sharp constants in weighted trace inequalities on Riemannian manifolds

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## Abstract

We establish some sharp weighted trace inequalities  $W^{1,2}(\rho^{1-2\sigma}, M) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(\partial M)$  on  $n + 1$  dimensional compact smooth manifolds with smooth boundaries, where  $\rho$  is a defining function of  $M$  and  $\sigma \in (0, 1)$ . This is stimulated by some recent work on fractional (conformal) Laplacians and related problems in conformal geometry, and also motivated by a conjecture of Aubin.

## 1 Introduction

Let  $\Omega$  be an open set in  $\mathbb{R}^n$ ,  $n \geq 1$ , and  $\rho(x) = \text{dist}(x, \partial\Omega)$  for  $x \in \Omega$ . There have been much work devoted to the structures of weighted Sobolev spaces of the type  $W^{k,p}(\rho^\alpha, \Omega)$  where  $\alpha \in \mathbb{R}$ ,  $k \in \mathbb{N}$  and  $1 \leq p \leq \infty$ , as well as to their applications in different areas such as (stochastic) partial differential equations and Riemannian manifolds with fractal boundaries or boundary singularities. We refer to the book [36] of Maz'ya and references therein for these topics.

In this paper, we would like to study sharp constants in weighted trace type inequalities  $W^{1,2}(\rho^{1-2\sigma}) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(\partial M)$  on Riemannian manifolds  $M$  with boundaries  $\partial M$ . Let us start from Euclidean spaces. Denote  $\dot{H}^\sigma(\mathbb{R}^n)$  as the  $\sigma$ -order homogeneous Sobolev space on  $\mathbb{R}^n$ ,  $n \geq 2$ , which is the closure of  $C_c^\infty(\mathbb{R}^n)$  under the norm

$$\|f\|_{\dot{H}^\sigma(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} f(x)|^2 dx \right)^{1/2}.$$

The sharp  $\sigma$ -order Sobolev inequality asserts that

$$\|f\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)}^2 \leq c(n, \sigma) \|f\|_{\dot{H}^\sigma(\mathbb{R}^n)}^2$$

for all  $f \in \dot{H}^\sigma(\mathbb{R}^n)$ , where

$$c(n, \sigma) = 2^{-2\sigma} \pi^{-\sigma} \left( \frac{\Gamma((n-2\sigma)/2)}{\Gamma((n+2\sigma)/2)} \right) \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2\sigma}{n}},$$

and the equality holds if and only if  $f(x)$  takes the form

$$c \left( \frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2\sigma}{2}}$$

for some  $c \in \mathbb{R}$ ,  $\lambda > 0$  and  $x_0 \in \mathbb{R}^n$ . These have been proved by Lieb in [34]. Set  $x = (x', x_{n+1}) \in \mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$  and

$$F(x', x_{n+1}) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x' - \xi, x_{n+1}) f(\xi) d\xi,$$

where

$$\mathcal{P}_\sigma(x', x_{n+1}) = \beta(n, \sigma) \frac{x_{n+1}^{2\sigma}}{(|x'|^2 + x_{n+1}^2)^{\frac{n+2\sigma}{2}}} \quad (1)$$

with the normalization constant  $\beta(n, \sigma) > 0$  such that  $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x', 1) dx' = 1$ . Then one has (see, e.g., [9])

$$N_\sigma \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2\sigma} |\nabla F(x', x_{n+1})|^2 dx = \|f\|_{\dot{H}^\sigma(\mathbb{R}^n)}^2,$$

where  $N_\sigma = 2^{2\sigma-1} \Gamma(\sigma) / \Gamma(1-\sigma)$ . Hence, we have

$$\|f\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)}^2 \leq S(n, \sigma) \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2\sigma} |\nabla F(x', x_{n+1})|^2 dx \quad (2)$$

for all  $f \in \dot{H}^\sigma(\mathbb{R}^n)$ , where  $S(n, \sigma) = N_\sigma \cdot c(n, \sigma)$ . Consequently, one can show (see, e.g., Proposition 2.1 below together with a density argument) that

$$\|U(\cdot, 0)\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)}^2 \leq S(n, \sigma) \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2\sigma} |\nabla U(x', x_{n+1})|^2 dx \quad (3)$$

for all  $U \in W^{1,2}(x_{n+1}^{1-2\sigma}, \mathbb{R}_+^{n+1})$ , which is the closure of  $C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$  under the norm

$$\|U\|_{W^{1,2}(x_{n+1}^{1-2\sigma}, \mathbb{R}_+^{n+1})} = \sqrt{\int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2\sigma} (|U|^2 + |\nabla U|^2) dx}.$$

Stimulated by several recent work on fractional (conformal) Laplacians and related problems in conformal geometry (see, e.g., [22, 10, 21, 26]) and a conjecture of Aubin [2], we study

weighted Sobolev trace inequalities of type (3) on Riemannian manifolds with boundaries. For  $n \geq 2$ , let  $(M, g)$  be an  $n + 1$  dimensional, compact, smooth Riemannian manifold with smooth boundary  $\partial M$ . We say a function  $\rho \in C^\infty(\overline{M})$  is a *defining function* of  $M$  if

$$\rho > 0 \quad \text{in } M, \quad \rho = 0 \text{ and } \nabla_g \rho \neq 0 \quad \text{on } \partial M.$$

Since  $\rho^{1-2\sigma}$ , where  $\sigma \in (0, 1)$  is a constant, belongs to the Muckenhoupt  $A_2$  class, we define the weighted Sobolev space  $H^1(\rho^{1-2\sigma}, M)$  as the closure of  $C^\infty(\overline{M})$  under the norm

$$\|u\|_{H^1(\rho^{1-2\sigma}, M)} = \left( \int_M \rho^{1-2\sigma} (|u|^2 + |\nabla u|^2) dv_g \right)^{\frac{1}{2}},$$

where  $dv_g$  denote the volume form of  $(M, g)$ .  $H^1(\rho^{1-2\sigma}, M)$  is a Hilbert space and it has a well-defined *trace operator*  $T$  (see, e.g., [36] or [39]) which continuously maps  $H^1(\rho^{1-2\sigma}, M)$  to  $H^\sigma(\partial M)$ , where  $H^\sigma(\partial M)$  is the  $\sigma$ -order Sobolev space on  $\partial M$ .

**Theorem 1.1.** *For  $n \geq 2$ , let  $(M, g)$  be an  $n + 1$  dimensional, compact, smooth Riemannian manifold with smooth boundary  $\partial M$ . Let  $\sigma \in (0, \frac{1}{2}]$ , and  $\rho$  be a defining function of  $M$  satisfying  $|\nabla_g \rho| = 1$  on  $\partial M$ . Then there exists a positive constant  $A = A(M, g, n, \rho, \sigma)$  such that*

$$\left( \int_{\partial M} |u|^{\frac{2n}{n-2\sigma}} ds_g \right)^{\frac{n-2\sigma}{n}} \leq S(n, \sigma) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + A \int_{\partial M} u^2 ds_g, \quad (4)$$

for all  $u \in H^1(\rho^{1-2\sigma}, M)$ , where  $ds_g$  denotes the induced volume form on  $\partial M$ .

For  $\sigma \in (\frac{1}{2}, 1)$ , we have

**Theorem 1.2.** *Let  $\sigma \in (\frac{1}{2}, 1)$ ,  $n \geq 4$  and  $(M, g)$  be an  $n + 1$  dimensional, compact, smooth Riemannian manifold with smooth boundary  $\partial M$ . Suppose in addition that  $\partial M$  is totally geodesic. Let  $\rho$  be a defining function of  $M$  satisfying  $\rho(x) = d(x) + O(d(x)^3)$  as  $d(x) \rightarrow 0$ , where  $d(x)$  denotes the distance between  $x$  and  $\partial M$  with respect to the metric  $g$ . Then there exists a positive constant  $A = A(M, g, n, \rho, \sigma)$  such that (4) holds for all  $u \in H^1(\rho^{1-2\sigma}, M)$ .*

**Remark 1.1.** *The constant  $S(n, \sigma)$  in (4) is optimal for all  $\sigma \in (0, 1)$ , see Proposition 2.2.*

**Remark 1.2.** *Theorem 1.2 may fail without any geometric assumption on  $\partial M$ . For example, it is the case when the mean curvature of  $\partial M$  is positive somewhere. In particular, (4) is false on any bounded smooth domain in  $\mathbb{R}^{n+1}$  when  $\sigma \in (1/2, 1)$ . However, Theorem 1.1 holds for all  $\sigma \in (0, 1)$  if  $S(n, \sigma)$  is replaced by any  $S > S(n, \sigma)$ , see Proposition 2.5.*

**Remark 1.3.** *It is clear that we only need to consider the case when  $M$  is connected. Throughout the paper, we assume this.*

When  $\sigma = \frac{1}{2}$ , (4) is a standard Sobolev trace inequality which has been extensively studied, see, e.g., Lions [35], Escobar [14], Beckner [5], Adimurthi-Yadava [1], Li-Zhu [32, 33] and many others. In particular, Li-Zhu [32] established Theorem 1.1 for  $\sigma = \frac{1}{2}$ . The sharp inequality (4) is in the same spirit of a conjecture posed by Aubin [2] which concerns the best constants in Sobolev embedding theorems on Riemannian manifolds. Aubin's conjecture had been confirmed through the work of Hebey-Vaugon [25], Aubin-Li [4] and Druet [11, 12]. Besides, various refinements of Aubin's conjecture were obtained in Druet-Hebey [13], Li-Ricciardi [31] and etc. These sharp Sobolev type inequalities play important roles in the study of nonlinear partial differential equations, see Aubin [3], Hebey [24], Schoen-Yau [42] and references therein.

For the defining function in the above theorems,  $(M, g/\rho^2)$  is *asymptotically hyperbolic* in the sense that  $(M, g/\rho^2)$  is a complete manifold and along any smooth curve in  $M \setminus \partial M$  tending to a point  $\xi \in \partial M$  all sectional curvatures of  $g/\rho^2$  approach to  $-1$  (see Mazzeo [37] or Mazzeo-Melrose [38]). On the conformal infinity  $(\partial M, [g|_{\partial M}])$  of  $(M, g/\rho^2)$ , one can define fractional order conformally invariant operators  $P_\sigma^g$  for  $\sigma \in (0, \frac{n}{2})$  except at most finite values, via normalized scattering operators (see Graham-Zworski [22] and Chang-González [10]), which leads to  $\sigma$ -scalar curvature  $R_\sigma^g := P_\sigma^g(1)$  on  $\partial M$ . A fractional Yamabe problem, which is to find a metric in  $[g|_{\partial M}]$  of constant  $\sigma$ -curvature and related ones, have been studied by Qing-Raske [41], González-Mazzeo-Sire [20] and González-Qing [21]. When  $\sigma \in (0, 1)$ , it can be formulated (see [21]) as seeking minimizers of the energy functional

$$I^\sigma[u] = \frac{N_\sigma \int_M \rho^{1-2\sigma} |\nabla u|^2 dv_g + \int_{\partial M} R_\sigma^g u^2 ds_g}{\left( \int_{\partial M} |u|^{\frac{2n}{n-2\sigma}} ds_g \right)^{\frac{n-2\sigma}{n}}}, \quad u \in H^1(\rho^{1-2\sigma}, M), \quad u \not\equiv 0 \text{ on } \partial M, \quad (5)$$

for some proper  $\rho$ . For  $\sigma = 1/2$ , it is the energy functional of a Yamabe problem with boundary initially studied by Escobar [15]. A fractional Nirenberg problem about prescribing  $\sigma$ -scalar curvature on  $\mathbb{S}^n$  has been studied by Jin-Li-Xiong [26, 27] and a fractional Yamabe flow has been studied by Jin-Xiong [28]. Variational problems related to energy functional (5) on bounded domains in Euclidean spaces have been studied by González [19], Palatucci-Sire [40].

Finally, we provide a brief sketch of the proofs of the two main theorems. Since the right hand side of (4) does not contain terms like  $\int_M \rho^{1-2\sigma} u^2 dv_g$ , we adapt a global argument from Li-Zhu [32, 33]. By contradiction, we assume that for any  $\alpha > 0$ ,

$$I_\alpha := \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + \alpha \int_{\partial M} |u|^2 ds_g}{\left( \int_{\partial M} |u|^{\frac{2n}{n-2\sigma}} ds_g \right)^{\frac{n-2\sigma}{n}}} < \frac{1}{S(n, \sigma)},$$

for some  $u \in H^1(\rho^{1-2\sigma}, M)$  with that  $u \not\equiv 0$  on  $\partial M$ . It follows that there exists a minimizer  $u_\alpha$  of  $I_\alpha$ , and  $u_\alpha$  blows up at exactly one point as  $\alpha \rightarrow \infty$ . One key step is the asymptotical analysis of  $u_\alpha$  near its blow up point. Here we have to overcome difficulties from the degeneracy and the lack of conformal invariance of the Euler-Lagrange equation of  $I_\alpha$  satisfied by  $u_\alpha$ . Another difference from [32] (the case  $\sigma = 1/2$ ) is that some Sobolev embedding theorems for  $H^1(\rho^{1-2\sigma}, M)$ ,

which play important roles in establishing the blow-up profile of  $u_\alpha$  in the interior of  $M$  in [32] in the case  $\sigma = \frac{1}{2}$ , fail when  $\sigma > \frac{1}{2}$  (see, e.g., Theorem 1 in page 135 or Corollary 2 in page 193 of [36]). However, we succeeded in establishing the optimal asymptotical behavior of  $u_\alpha$  on the boundary  $\partial M$  (Proposition 3.3). In this step, a Liouville type theorem in Jin-Li-Xiong [26] and *Neumann functions* for degenerate equations in Theorem 1.3 are used. The last step is to derive a contradiction by checking balance via a Pohozaev type inequality in some proper region, where a Harnack inequality established by Cabre-Sire [8] or Tan-Xiong [43] is used to obtain the asymptotical behavior of  $u_\alpha$  near its blowup point in  $M$  from that on  $\partial M$ . Some extra arguments on  $\partial M$  are needed for  $\sigma > \frac{1}{2}$ .

**Theorem 1.3.** *Let  $f \in L^1(\partial M)$  with mean value zero, i.e.,  $\int_{\partial M} f = 0$ . Then there exists a weak solution  $u \in W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M)$  of (59) where  $\varepsilon_0 > 0$  depending only on  $n$  and  $\sigma$ . Consequently, if  $f = \delta_{x_0} - \frac{1}{|\partial M|_g}$  for some  $x_0 \in \partial M$ , where  $\delta_{x_0}$  is the delta function at  $x_0$  and  $|\partial M|_g$  is the area of  $\partial M$  with respect to the induced metric  $g$ , then there exists a weak solution  $u \in W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M) \cap H_{loc}^1(\rho^{1-2\sigma}, \overline{M} \setminus \{x_0\})$  of (59) with mean value zero. Moreover, for all  $x \in \overline{M} \setminus \{x_0\}$ ,*

$$A_1 \text{dist}_g(x, x_0)^{2\sigma-n} - A_0 \leq u(x) \leq A_2 \text{dist}_g(x, x_0)^{2\sigma-n}$$

where  $A_0, A_1, A_2$  are positive constants depending only on  $M, g, n, \sigma, \rho$ .

The proof of Theorem 1.3 follows from Lemma A.5, Theorem A.5 and some approximation arguments. When  $\sigma = 1/2$ , Theorem 1.3 follows directly from Brezis-Strauss [7] and Kenig-Pipher [29].

**Notations.** We collect below a list of the main notations used throughout the paper.

- We always assume that  $n \geq 2, \sigma \in (0, 1)$ , and  $\rho$  is a smooth defining function as in Theorem 1.1 without otherwise stated. Denote  $q = \frac{2n}{n-2\sigma}$ .
- For a domain  $D \subset \mathbb{R}^{n+1}$  with boundary  $\partial D$ , we denote  $\partial' D$  as the interior of  $\overline{D} \cap \partial \mathbb{R}_+^{n+1}$  in  $\mathbb{R}^n = \partial \mathbb{R}_+^{n+1}$  and  $\partial'' D = \partial D \setminus \partial' D$ .
- For  $\bar{x} \in \mathbb{R}^{n+1}$ ,  $\mathcal{B}_r(\bar{x}) := \{x \in \mathbb{R}^{n+1} : |x - \bar{x}| = \sqrt{(x_1 - \bar{x}_1)^2 + \cdots + (x_{n+1} - \bar{x}_{n+1})^2} < r\}$ ,  $\mathcal{B}_r^+(\bar{x}) := \mathcal{B}_r(\bar{x}) \cap \mathbb{R}_+^{n+1}$ . If  $\bar{x} \in \partial \mathbb{R}_+^{n+1}$ ,  $B_r(\bar{x}) := \{x = (x', 0) : |x' - \bar{x}'| < r\}$ . Hence  $\partial' \mathcal{B}_r^+(\bar{x}) = B_r(\bar{x})$  if  $\bar{x} \in \partial \mathbb{R}_+^{n+1}$ . We will not keep writing the center  $\bar{x}$  if  $\bar{x} = 0$ .

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## 2 Preliminaries

**Proposition 2.1.** *For any  $u \in C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$ , we have*

$$\left( \int_{\mathbb{R}^n} |u(x', 0)|^q dx' \right)^{\frac{2}{q}} \leq S(n, \sigma) \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2\sigma} |\nabla u(x)|^2 dx.$$

Moreover, the above inequality fails if  $S(n, \sigma)$  is replaced by any smaller constant.

*Proof.* It follows from (3) and Lemma A.3 of [26]. See also Corollary 5.3 of [21].  $\square$

**Proposition 2.2.** *Let  $M$  be as in Theorem 1.1. Let  $\sigma \in (0, 1)$ , and  $\rho$  be a defining function of  $\partial M$  with  $|\nabla_g \rho| = 1$  on  $\partial M$ . Suppose there exist some positive constants  $\tilde{S}$  and  $\tilde{A}$  such that, for all  $u \in H^1(\rho^{1-2\sigma}, M)$ ,*

$$\left( \int_{\partial M} |u|^q ds_g \right)^{\frac{2}{q}} \leq \tilde{S} \int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + \tilde{A} \int_{\partial M} |u|^2 ds_g.$$

Then  $\tilde{S} \geq S(n, \sigma)$ .

*Proof.* Given Proposition 2.1, the proof is standard (see, e.g., Proposition 4.2 of [24]). We include it here for completeness and to illustrate the role of  $|\nabla \rho| = 1$ . We argue by contradiction. Suppose that there exists a Riemannian manifold  $(M, g)$ , a defining function  $\rho$  of  $\partial M$  with  $|\nabla_g \rho| = 1$  on  $\partial M$ ,  $\sigma \in (0, 1)$ ,  $\tilde{S} < S(n, \sigma)$  and  $\tilde{A} > 0$  such that for all  $u \in H^1(\rho^{1-2\sigma}, M)$ ,

$$\left( \int_{\partial M} |u|^q ds_g \right)^{\frac{2}{q}} \leq \tilde{S} \int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + \tilde{A} \int_{\partial M} |u|^2 ds_g. \quad (6)$$

Let  $x \in \partial M$ . For any  $\varepsilon > 0$ , which will be chosen sufficiently small, there exists a chart  $(\Omega, \varphi)$  of  $M$  at  $x$  and  $\delta > 0$  such that  $\varphi(\Omega) = \mathcal{B}_\delta^+(0)$  the upper half Euclidean ball of center 0 and radius  $\delta$  in  $\mathbb{R}_+^{n+1}$ , and

$$(1 - \varepsilon)\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)\delta_{ij}. \quad (7)$$

By assumption, (6) holds for any  $u \in C_c^\infty(\Omega \cup (\partial\Omega \cap \partial M))$ , i.e.,

$$\begin{aligned} \left( \int_{B_\delta(0)} |u|^q \sqrt{\det(g_{ij})} dx' \right)^{\frac{2}{q}} &\leq \tilde{S} \int_{\mathcal{B}_\delta^+(0)} \rho^{1-2\sigma} g^{ij} u_i u_j \sqrt{\det(g_{ij})} dx \\ &\quad + \tilde{A} \int_{B_\delta(0)} |u|^2 \sqrt{\det(g_{ij})} dx'. \end{aligned}$$

It follows from (7),  $|\nabla_g \rho| = 1$  and  $\rho = 0$  on  $\partial M$  that there exists  $\delta_0 > 0$ ,  $\tilde{S}' < S(n, \sigma)$ ,  $\tilde{A}' > 0$  such that for all  $\delta \in (0, \delta_0)$  and  $u \in C_c^\infty(\mathcal{B}_\delta(0) \cup B_\delta(0))$ , i.e.,

$$\left( \int_{B_\delta(0)} |u|^q dx' \right)^{\frac{2}{q}} \leq \tilde{S}' \int_{\mathcal{B}_\delta^+(0)} x_{n+1}^{1-2\sigma} |\nabla u|^2 dx + \tilde{A}' \int_{B_\delta(0)} |u|^2 dx'.$$

By Hölder's inequality,  $\int_{B_\delta(x)} |u|^2 dx' \leq |B_\delta(0)|^{\frac{q-2}{q}} \left( \int_{B_\delta(0)} |u|^q dx' \right)^{\frac{2}{q}}$ . By choosing  $\delta$  sufficiently small, we have that there exists  $\tilde{S}'' < S(n, \sigma)$  such that for all  $u \in C_c^\infty(\mathcal{B}_\delta(0) \cup B_\delta(0))$

$$\left( \int_{B_\delta(0)} |u|^q dx' \right)^{\frac{2}{q}} \leq \tilde{S}'' \int_{\mathcal{B}_\delta^+(0)} x_{n+1}^{1-2\sigma} |\nabla u|^2 dx.$$

Consequently, by a scaling argument, we have

$$\left( \int_{\mathbb{R}^n} |u(x', 0)|^q dx' \right)^{\frac{2}{q}} \leq \tilde{S}'' \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2\sigma} |\nabla u(x)|^2 dx.$$

for any  $u \in C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$ , which contradicts Proposition 2.1.  $\square$

**Proposition 2.3.** *Assume the assumptions in Proposition 2.2. Then for any  $\varepsilon > 0$  there exists a positive constant  $B_\varepsilon$  such that*

$$\left( \int_{\partial M} |u|^q ds_g \right)^{\frac{2}{q}} \leq (S(n, \sigma) + \varepsilon) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + B_\varepsilon \int_M \rho^{1-2\sigma} |u|^2 dv_g.$$

*Proof.* It also follows from Proposition 2.1 and a standard partition of unity argument, see, e.g., Theorem 4.5 of [24] on page 95.  $\square$

For every  $\alpha > 0$ , consider the functional

$$I_\alpha[u] = \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + \alpha \int_{\partial M} |u|^2 ds_g}{\left( \int_{\partial M} |u|^q ds_g \right)^{2/q}}, \quad u \in H^1(\rho^{1-2\sigma}, M), \quad u \not\equiv 0 \text{ on } \partial M.$$

**Proposition 2.4.** *Suppose that for some  $\alpha > 0$ ,*

$$\xi_\alpha := \inf_{u \in H^1(\rho^{1-2\sigma}, M), u|_{\partial M} \not\equiv 0} I_\alpha[u] < \frac{1}{S(n, \sigma)}, \quad (8)$$

*then  $\xi_\alpha$  is achieved by a nonnegative function  $u_\alpha \in H^1(\rho^{1-2\sigma}, M)$  with*

$$\int_{\partial M} u_\alpha^q ds_g = 1. \quad (9)$$

*Proof.* Given Proposition 2.3, the Proposition follows from standard calculus of variations, see page 452 of [32].  $\square$

**Proposition 2.5.** *Assume the assumptions in Proposition 2.2. For any  $\varepsilon > 0$ , there exists a positive constant  $A_\varepsilon$  such that*

$$\left( \int_{\partial M} |u|^q ds_g \right)^{\frac{2}{q}} \leq (S(n, \sigma) + \varepsilon) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 dv_g + A_\varepsilon \int_{\partial M} |u|^2 ds_g.$$

*Proof.* Given Propositions 2.3 and 2.4, and Corollary A.1, the proof of Proposition 2.5 is similar to Proposition 1.2 of [32] and we omit it here.  $\square$

### 3 Asymptotic analysis

For brevity, from now on we write  $S$  instead of  $S(n, \sigma)$ . We prove Theorem 1.1 by contradiction. Namely, assume that for any  $\alpha \geq 1$ ,

$$\xi_\alpha < \frac{1}{S}, \quad (10)$$

where  $\xi_\alpha$  is defined as in Proposition 2.4. Let  $u_\alpha$  be some nonnegative minimizer of  $I_\alpha$  obtained in Proposition 2.4 which satisfies

$$\xi_\alpha = \int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 dv_g + \alpha \int_{\partial M} u_\alpha^2 ds_g, \quad \int_{\partial M} u_\alpha^q ds_g = 1, \quad (11)$$

and for any  $\varphi \in H^1(\rho^{1-2\sigma}, M)$ ,

$$\int_M \rho^{1-2\sigma} \langle \nabla_g u_\alpha, \nabla_g \varphi \rangle_g dv_g + \alpha \int_{\partial M} u_\alpha \varphi ds_g = \xi_\alpha \int_{\partial M} u_\alpha^{q-1} \varphi ds_g. \quad (12)$$

The geodesic distance function  $d(x) := \text{dist}(x, \partial M)$  determines for some  $\varepsilon_0 > 0$  an identification of  $\partial M \times [0, \varepsilon_0)$  with a neighborhood of  $\partial M$  in  $M$ :  $(x', d) \in \partial M \times [0, \varepsilon_0)$  corresponds to the point obtained by following the integral curve of  $\nabla_g d$  emanating from  $x'$  for  $d$  units of time. Furthermore,  $\nabla_g d$  is orthogonal to the slices  $\partial M \times \{d\}$ . Define  $\nu := -\nabla_g d$  for  $d < \varepsilon_0$ . It follows from Theorem A.2, Theorem A.3 and Proposition A.1 that  $u_\alpha \in C^\gamma(\overline{M}) \cap C^\infty(M) \cap C^\infty(\partial M)$  for some  $\gamma \in (0, 1)$  and  $\rho^{1-2\sigma} \frac{\partial_g u_\alpha}{\partial \nu} \in C(\partial M \times [0, \varepsilon_0/2])$ . Hence,  $u_\alpha$  satisfies the Euler-Lagrange equation

$$\begin{cases} \text{div}_g \left( \rho^{1-2\sigma} \nabla_g u_\alpha \right) = 0, & \text{in } M, \\ \lim_{d \rightarrow 0} \rho^{1-2\sigma}(x', d) \frac{\partial_g u_\alpha}{\partial \nu}(x', \rho) = \xi_\alpha u_\alpha^{q-1}(x') - \alpha u_\alpha(x'), & \text{on } \partial M. \end{cases} \quad (13)$$

in the pointwise sense.

It follows from the maximum principle that  $\max_{\overline{M}} u_\alpha = \max_{\partial M} u_\alpha$ . Let  $u_\alpha(x_\alpha) = \max_{\overline{M}} u_\alpha$ , where  $x_\alpha \in \partial M$ , and  $\mu_\alpha = u_\alpha(x_\alpha)^{-\frac{2}{n-2\sigma}}$ . By a Hopf Lemma (see, e.g., Proposition 4.11 in [8]), we have  $\xi_\alpha u_\alpha(x_\alpha)^{q-1} - \alpha u_\alpha(x_\alpha) > 0$ , that is

$$\alpha \mu_\alpha^{2\sigma} < \xi_\alpha. \quad (14)$$

Hence,  $\lim_{\alpha \rightarrow \infty} \mu_\alpha^{2\sigma} = 0$ .

**Lemma 3.1.** *As  $\alpha \rightarrow \infty$ , we have*

$$\xi_\alpha \rightarrow \frac{1}{S}, \quad (15a)$$

$$\alpha \|u_\alpha\|_{L^2(\partial M)}^2 \rightarrow 0. \quad (15b)$$

*Proof.* For all small  $\varepsilon > 0$ , it follows from Proposition 2.5 that

$$\begin{aligned} 1 &\leq (S + \varepsilon) \int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 dv_g + A_\varepsilon \int_{\partial M} u_\alpha^2 ds_g \\ &= (S + \varepsilon) \xi_\alpha + (A_\varepsilon - (S + \varepsilon) \alpha) \int_{\partial M} u_\alpha^2 ds_g. \end{aligned}$$

Hence, for every  $\alpha \geq \frac{2A_\varepsilon}{S+\varepsilon}$  we have

$$\frac{1}{S + \varepsilon} \leq \xi_\alpha < \frac{1}{S}, \quad \frac{S}{2} \alpha \int_{\partial M} u_\alpha^2 ds_g < \frac{\varepsilon}{S}.$$

(15a) and (15b) follow immediately.  $\square$

Let  $x = (x_1, \dots, x_n, x_{n+1}) = (x', x_{n+1})$  be *Fermi coordinates* (see, e.g., [15]) at  $x_\alpha$ , where  $(x_1, \dots, x_n)$  are normal coordinates on  $\partial M$  at  $x_\alpha$  and  $\gamma(x_{n+1})$  is the geodesic leaving from  $(x_1, \dots, x_n)$  in the orthogonal direction to  $\partial M$  and parametrized by arc length. In this coordinate system,

$$\sum_{1 \leq i, j \leq n+1} g_{ij}(x) dx_i dx_j = dx_{n+1}^2 + \sum_{1 \leq i, j \leq n} g_{ij}(x) dx_i dx_j.$$

Moreover,  $g^{ij}$  has the following Taylor expansion near  $\partial M$ :

**Lemma 3.2** (Lemma 3.2 in [15]). *For  $\{x_k\}_{k=1, \dots, n+1}$  are small,*

$$g^{ij}(x) = \delta^{ij} + 2h^{ij}(x', 0)x_{n+1} + O(|x|^2), \quad (16)$$

where  $i, j = 1, \dots, n$  and  $h_{ij}$  is the second fundamental form of  $\partial M$ .

For suitably small  $\delta_0 > 0$  (independent of  $\alpha$ ), we define  $v_\alpha$  in a neighborhood of  $x_\alpha = 0$  by

$$v_\alpha(x) = \mu_\alpha^{(n-2\sigma)/2} u_\alpha(\mu_\alpha x), \quad x \in \mathcal{B}_{\delta_0/\mu_\alpha}^+.$$

It follows that

$$\begin{cases} \operatorname{div}_{g_\alpha} \left( \rho_\alpha^{1-2\sigma} \nabla_{g_\alpha} v_\alpha \right) = 0, & \text{in } \mathcal{B}_{\delta_0/\mu_\alpha}^+ \\ \lim_{x_{n+1} \rightarrow 0^+} \rho_\alpha^{1-2\sigma} \frac{\partial_{g_\alpha} v_\alpha}{\partial \nu} = \xi_\alpha v_\alpha^{q-1} - \alpha \mu_\alpha^{2\sigma} v_\alpha, & \text{on } \partial' \mathcal{B}_{\delta_0/\mu_\alpha}^+ = B_{\delta_0/\mu_\alpha} \\ v_\alpha(0) = 1, \quad 0 \leq v_\alpha \leq 1, \end{cases} \quad (17)$$

where  $g_\alpha(x) = g_{ij}(\mu_\alpha x) dx_i dx_j$ ,  $\rho_\alpha(x) = \rho(\mu_\alpha x)/\mu_\alpha$ . It follows from (14) and Theorem A.2 in the Appendix that for all  $R > 1$ ,

$$\|v_\alpha\|_{C^\gamma(\mathcal{B}_R^+)} + \|v_\alpha\|_{H^1(\rho_\alpha^{1-2\sigma}, \mathcal{B}_R^+)} \leq C(R), \quad \text{for all sufficiently large } \alpha, \quad (18)$$

where  $\gamma \in (0, 1)$  is independent of  $R$  and  $\alpha$ . It follows that there exists  $v \in C_{loc}^\gamma(\overline{\mathbb{R}_+^{n+1}}) \cap H_{loc}^1(x_{n+1}^{1-2\sigma}, \overline{\mathbb{R}_+^{n+1}})$  such that along some subsequence,

$$\begin{cases} v_\alpha \rightarrow v \text{ in } C^{\gamma/2}(\mathcal{B}_R^+), \\ v_\alpha \rightharpoonup v \text{ weakly in } H^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_R^+) \end{cases} \quad (19)$$

for any  $R > 0$  as  $\alpha \rightarrow \infty$ . Since  $v_\alpha(0) = 1$ , we have

$$\begin{aligned} \int_{B_1} v_\alpha^q ds_{g_\alpha} &\geq 1/C > 0, \\ \int_{B_1} v_\alpha^2 ds_{g_\alpha} &\geq 1/C > 0. \end{aligned} \quad (20)$$

On the other hand,

$$\alpha \|u_\alpha\|_{L^2(\partial M)}^2 \geq \alpha \int_{B_{\mu_\alpha}(x_\alpha)} u_\alpha^2 = \alpha \mu_\alpha^{2\sigma} \int_{B_1} v_\alpha^2,$$

where we abused notation by denoting  $B_r(x_\alpha)$  as the geodesic ball on  $\partial M$  centered at  $x_\alpha$  with radius  $r$ . It follows from (15b) and (20) that

$$\lim_{\alpha \rightarrow \infty} \alpha \mu_\alpha^{2\sigma} = 0. \quad (21)$$

From (17), (21) and (15a), we conclude that  $v$  is a weak solution (see Section A.2 for the definition of weak solutions) of

$$\begin{cases} \operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v) = 0, & \text{in } \mathbb{R}_+^{n+1}, \\ - \lim_{x_{n+1} \rightarrow 0^+} x_{n+1}^{1-2\sigma} \partial_{x_{n+1}} v = \frac{1}{S} v^{q-1}, & \text{on } \partial \mathbb{R}_+^{n+1}, \\ v(0) = 1, \quad 0 \leq v \leq 1. \end{cases} \quad (22)$$

By a Liouville type theorem, Theorem 1.5 in [26],

$$v(x', 0) = \left( \frac{1}{1 + \tilde{c}(n, \sigma)|x'|^2} \right)^{\frac{n-2\sigma}{2}}, v(x', x_{n+1}) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x' - y', x_{n+1}) v(y', 0) dy',$$

where  $\tilde{c}(n, \sigma)$  is a positive constant such that  $\int_{\mathbb{R}^n} v^q(z) dz = 1$ , and  $\mathcal{P}_\sigma(x)$  is given in (1). Due to the uniqueness of the limit function  $v$ , we know that (19) holds for all  $\alpha \rightarrow \infty$ .

**Proposition 3.1.** *For  $\delta_0 = \delta_0(M, g) > 0$  small enough,*

$$\lim_{\alpha \rightarrow \infty} \int_{B_{\delta_0/\mu_\alpha}} |v_\alpha - v|^q = 0.$$

*Proof.* Note that  $v_\alpha \geq 0$  and

$$\int_{B_{\delta_0/\mu_\alpha}} v_\alpha^q \leq \int_{\partial M} u_\alpha^q = 1. \quad (23)$$

For any  $\varepsilon > 0$ , choose  $R > 0$  such that  $\int_{\mathbb{R}^n \setminus B_R} v^q(x', 0) dx' \leq \varepsilon$ . It follows from (19) that  $\int_{B_R} |v_\alpha - v|^q \leq \varepsilon$  and  $1 - \int_{B_R} v_\alpha^q < 2\varepsilon$  for all  $\alpha$  sufficiently large. Then

$$\begin{aligned} & \int_{B_{\delta_0/\mu_\alpha}} |v_\alpha - v|^q \\ &= \int_{B_{\delta_0/\mu_\alpha} \cap B_R} |v_\alpha - v|^q + \int_{B_{\delta_0/\mu_\alpha} \cap B_R^c} |v_\alpha - v|^q \\ &\leq \int_{B_{\delta_0/\mu_\alpha} \cap B_R} |v_\alpha - v|^q + 2^q \int_{B_{\delta_0/\mu_\alpha} \cap B_R^c} v_\alpha^q + 2^q \int_{B_{\delta_0/\mu_\alpha} \cap B_R^c} v^q \\ &\leq \varepsilon + 2^q (1 - \int_{B_R} v_\alpha^q) + 2^q (1 - \int_{B_R} v^q) \leq \varepsilon (1 + 3 \cdot 2^q), \end{aligned}$$

which finishes the proof.  $\square$

**Corollary 3.1.** *For all  $\delta_1 > 0$  we have*

$$\lim_{\alpha \rightarrow \infty} \int_{B_{\delta_1}(x_\alpha) \cap \partial M} u_\alpha^q = 1.$$

*Proof.* It follows immediately from Proposition 3.1.  $\square$

Let  $\tilde{G}_\alpha$  be the weak solution of

$$\begin{cases} -\operatorname{div}_g(\rho^{1-2\sigma} \nabla_g \tilde{G}_\alpha) = 0, & \text{in } M, \\ \lim_{y \rightarrow x \in \partial M} \rho^{1-2\sigma}(y) \frac{\partial}{\partial \nu} \tilde{G}_\alpha(y) = \delta_{x_\alpha} - \frac{1}{|\partial M|_g}, & \text{on } \partial M, \end{cases}$$

constructed in Theorem A.5. We can find a positive constant  $C > 0$  sufficiently large depending only on  $M, g, n, \sigma, \rho$  such that  $G_\alpha := \tilde{G}_\alpha + C \geq 1$  on  $\overline{M}$ .

**Proposition 3.2.** Let  $\varphi_\alpha(x) = \mu_\alpha^{\frac{n-2\sigma}{2}} G_\alpha(x)$ ,  $\tilde{g}_{ij} = \varphi_\alpha^{\frac{4}{n-2\sigma}} g_{ij}$  and  $a = 2 - \frac{2(n-1)}{n-2\sigma}$ . Then  $w_\alpha := \frac{u_\alpha}{\varphi_\alpha}$  satisfies

$$\begin{cases} \operatorname{div}_{\tilde{g}} \left( \varphi_\alpha^a \rho^{1-2\sigma} \nabla_{\tilde{g}} w_\alpha \right) = 0, & \text{in } M, \\ \lim_{y \rightarrow \bar{x} \in \partial M} \varphi_\alpha^a \rho^{1-2\sigma} \frac{\partial_{\tilde{g}} w_\alpha(y)}{\partial \tilde{\nu}} \leq \xi_\alpha w_\alpha^{q-1}(\bar{x}), & \bar{x} \in \partial M \setminus \{x_\alpha\}, \end{cases} \quad (24)$$

for  $\alpha \geq \frac{1}{|\partial M|_g}$ .

*Proof.* The proof follows from some direct computations. For brevity, we drop the subscript  $\alpha$  of  $\varphi_\alpha$  and  $u_\alpha$ . First of all,

$$\begin{aligned} & \operatorname{div}_{\tilde{g}} \left( \varphi^a \rho^{1-2\sigma} \nabla_{\tilde{g}} \frac{u}{\varphi} \right) \\ &= \varphi^{a-1-\frac{4}{n-2\sigma}} \operatorname{div}_g \left( \rho^{1-2\sigma} \nabla_g u \right) - u \varphi^{a-2-\frac{4}{n-2\sigma}} \operatorname{div}_g \left( \rho^{1-2\sigma} \nabla_g \varphi \right) \\ & \quad + \left( a - 2 + \frac{2(n-1)}{n-2\sigma} \right) \rho^{1-2\sigma} \varphi^{a-2-\frac{4}{n-2\sigma}} \left( \langle \nabla_g u, \nabla_g \varphi \rangle_g - u \varphi |\nabla_g \varphi|_g^2 \right) \\ &= 0. \end{aligned}$$

On the other hand, in Fermi coordinate system centered at  $\bar{x}$ ,

$$\begin{aligned} & \lim_{x_{n+1} \rightarrow 0} \varphi^a \rho^{1-2\sigma} \frac{\partial_{\tilde{g}}}{\partial \tilde{\nu}} \left( \frac{u}{\varphi} \right) \\ &= \lim_{x_{n+1} \rightarrow 0} \varphi^a \rho^{1-2\sigma} \left( \frac{1}{\varphi} \frac{\partial u}{\partial x_{n+1}} - \frac{u}{\varphi^2} \frac{\partial \varphi}{\partial x_{n+1}} \right) \tilde{g}^{n+1, n+1} \left\langle \frac{\partial}{\partial x_{n+1}}, \tilde{\nu} \right\rangle_{\tilde{g}} \\ &= \varphi^{a-1-\frac{2}{n-2\sigma}} \left( \xi_\alpha u^{\frac{n+2\sigma}{n-2\sigma}} - \alpha u \right) + \varphi^{a-2-\frac{2}{n-2\sigma}} u \mu_\alpha^{\frac{n-2\sigma}{2}} \frac{1}{|\partial M|} \\ &\leq \xi_\alpha \left( \frac{u}{\varphi} \right)^{\frac{n+2\sigma}{n-2\sigma}} + \varphi^{a-2-\frac{2}{n-2\sigma}} u \mu_\alpha^{\frac{n-2\sigma}{2}} \left( \frac{1}{|\partial M|_g} - \alpha \right) \\ &\leq \xi_\alpha \left( \frac{u}{\varphi} \right)^{\frac{n+2\sigma}{n-2\sigma}}, \end{aligned}$$

provided  $\alpha \geq \frac{1}{|\partial M|_g}$ . □

**Proposition 3.3.** Suppose the assumptions in Proposition 3.2. Then there exists some constant  $C$  depending only on  $M, g, n, \rho, \sigma$  such that for all  $\alpha \geq 1$ ,

$$w_\alpha \leq C, \quad \text{on } \partial M.$$

*Proof.* In the following,  $C$  denotes some constant which may depend on  $M, g, n, \rho, \sigma$  but not on  $\alpha$  and may vary from line to line.

It suffices to prove the proposition for large  $\alpha$ , in particular, say,  $\alpha \geq \max\{\frac{1}{|\partial M|_g}, 1\}$ . Let  $\tilde{\rho} := \varphi_\alpha^{\frac{2}{n-2\sigma}} \rho$ . Then (24) can be rewritten as

$$\begin{cases} \operatorname{div}_{\tilde{g}} \left( \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} w_\alpha \right) = 0, & \text{in } M, \\ \lim_{y \rightarrow \bar{x}} \tilde{\rho}^{1-2\sigma} \frac{\partial_{\tilde{g}} w_\alpha(y)}{\partial \tilde{\nu}} \leq \xi_\alpha w_\alpha^{q-1}(\bar{x}), & \text{for } \bar{x} \in \partial M \setminus \{x_\alpha\}, \end{cases} \quad (25)$$

where the limit is taken in the sense explained in the paragraph above (13). In the following, we shall abuse notation a little by writing  $\psi^{-1}(\mathcal{B}_\delta^+(0))$  as  $\mathcal{B}_\delta^+(0)$  where  $(\psi^{-1}(\mathcal{B}_\delta^+(0)), \psi)$  is a Fermi coordinate of  $M$  at  $x_\alpha$ , and denoting  $B_\delta(x_\alpha)$  as the geodesic ball on  $\partial M$  centered at  $x_\alpha$  with radius  $\delta$  as before. Note that the interior of  $\overline{\mathcal{B}_\delta^+(0)} \cap \partial M$  is  $B_\delta(x_\alpha)$ .

**Step 1.** We claim that there exist some constants  $0 < \delta_2 \ll 1$ ,  $s_0 > q$  independent of  $\alpha$  such that

$$\int_{\partial M \setminus B_{\mu_\alpha/\delta_2}(x_\alpha)} w_\alpha^{s_0} ds_{\tilde{g}} \leq C. \quad (26)$$

For any  $\varepsilon > 0$ , it follows from Proposition 3.1 that there exists a small  $\delta_2$  such that

$$\begin{aligned} \int_{\partial M \setminus B_{\mu_\alpha/\delta_2}(x_\alpha)} w_\alpha^q ds_{\tilde{g}} &= \int_{\partial M \setminus B_{\mu_\alpha/\delta_2}(x_\alpha)} u_\alpha^q ds_g \\ &= 1 - \int_{\partial' \mathcal{B}_{1/\delta_2}^+} v_\alpha^q \\ &\leq \varepsilon. \end{aligned} \quad (27)$$

Without loss of generality, we may assume  $10\mu_\alpha/\delta_2 < \delta_0$  where  $\delta_0$  is the constant such that the Fermi coordinate system centered at  $x_\alpha$  exists in  $\mathcal{B}_{\delta_0}^+(x_\alpha)$ .

We choose  $\eta$  to be some cutoff function satisfying

$$\begin{aligned} \eta(x) &= 1 \text{ if } |x| \geq \mu_\alpha/\delta_2, \quad \eta(x) = 0 \text{ if } |x| \leq \mu_\alpha/(2\delta_2), \\ \text{and } \eta &= \eta(|x|) \text{ in the Fermi coordinate system centered at } x_\alpha. \end{aligned}$$

Multiplying (25) by  $w_\alpha^k \eta^2$  for  $k > 1$  and integrating by parts, we obtain

$$\int_M \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} w_\alpha \nabla_{\tilde{g}} (w_\alpha^k \eta^2) dv_{\tilde{g}} \leq \xi_\alpha \int_{\partial M} w_\alpha^{q-1+k} \eta^2 ds_{\tilde{g}}.$$

By a direct computation, we see that

$$\begin{aligned}
& \int_M \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} w_\alpha \nabla_{\tilde{g}} (w_\alpha^k \eta^2) \, dv_{\tilde{g}} \\
&= \frac{4k}{(k+1)^2} \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^{(k+1)/2} \eta)|^2 \, dv_{\tilde{g}} + \frac{k-1}{(k+1)^2} \int_M w_\alpha^{k+1} \operatorname{div}_{\tilde{g}} \left( \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta^2 \right) \, dv_{\tilde{g}} \\
&\quad - \frac{4k}{(k+1)^2} \int_M \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} |\nabla_{\tilde{g}} \eta|^2 \, dv_{\tilde{g}},
\end{aligned}$$

where we have used that  $\lim_{\rho \rightarrow 0} \tilde{\rho}^{1-2\sigma} \frac{\partial_{\tilde{g}} \eta^2}{\partial \nu} = 0$  since  $\eta$  is radial. In conclusion, we obtain

$$\begin{aligned}
& \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^{(k+1)/2} \eta)|^2 \, dv_{\tilde{g}} \\
&\leq -\frac{k-1}{4k} \int_M w_\alpha^{k+1} \operatorname{div}_{\tilde{g}} \left( \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta^2 \right) \, dv_{\tilde{g}} + \int_M \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} |\nabla_{\tilde{g}} \eta|^2 \, dv_{\tilde{g}} \\
&\quad + \frac{\xi_\alpha (k+1)^2}{4k} \int_{\partial M} w_\alpha^{q-1+k} \eta^2 \, ds_{\tilde{g}}.
\end{aligned} \tag{28}$$

Since  $\tilde{g}^{ij} \sim \mu_\alpha^2 \delta^{ij}$  in  $\mathcal{B}_{2\mu_\alpha/\delta_2}^+(x_\alpha) \setminus \mathcal{B}_{\mu_\alpha/(4\delta_2)}^+(x_\alpha)$ , we have

$$|\nabla_{\tilde{g}} \eta| + |\nabla_{\tilde{g}}^2 \eta| \leq C.$$

Since  $\eta$  is radial in the Fermi coordinate system, using (65a), (65b) and (65c), we have

$$|\operatorname{div}_{\tilde{g}} (\tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta^2)| \leq C \tilde{\rho}^{1-2\sigma}.$$

Taking  $1 < k \leq q-1$  in (28) and using Theorem A.1 and Theorem A.5, it follows that

$$\begin{aligned}
& \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^{(k+1)/2} \eta)|^2 \, dv_{\tilde{g}} \\
&\leq C(k, \delta_2) + \frac{\xi_\alpha (k+1)^2}{4k} \int_{\partial M} w_\alpha^{q-1+k} \eta^2 \, ds_{\tilde{g}} \\
&\leq C(k, \delta_2) + \frac{\xi_\alpha (k+1)^2}{4k} \varepsilon^{(q-2)/q} \left( \int_{\partial M} (w_\alpha^{(1+k)/2} \eta)^q \, ds_{\tilde{g}} \right)^{2/q} \\
&\leq C(k, \delta_2) + C \varepsilon^{(q-2)/q} \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^{(k+1)/2} \eta)|^2 \, dv_{\tilde{g}},
\end{aligned}$$

where we used

$$\begin{aligned}
& \int_{M \cap (\mathcal{B}_{\mu_\alpha/\delta_2}^+ \setminus \mathcal{B}_{\mu_\alpha/(2\delta_2)}^+)} \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} dv_{\tilde{g}} \\
& \leq C(\delta_2) \int_{M \cap (\mathcal{B}_{\mu_\alpha/\delta_2}^+ \setminus \mathcal{B}_{\mu_\alpha/(2\delta_2)}^+)} \left(\frac{\rho}{\mu_\alpha}\right)^{1-2\sigma} (\mu_\alpha^{(n-2\sigma)/2} u_\alpha)^{k+1} \mu_\alpha^{-(n+1)} dv_g \\
& \leq C(\delta_2) \int_{1/(2\delta_2) \leq |z| \leq 1/\delta_2} \rho_\alpha(z)^{1-2\sigma} v_\alpha(z)^{k+1} dv_{g_\alpha} \quad \text{by changing variables} \\
& \leq C(k, \delta_2),
\end{aligned} \tag{29}$$

and  $\rho_\alpha(z)$ ,  $v_\alpha(z)$  are those in (17).

Taking  $\varepsilon > 0$  sufficiently small, we have

$$\int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}}(w_\alpha^{(k+1)/2} \eta)|^2 dv_{\tilde{g}} \leq C.$$

The claim follows immediately from Theorem A.1 in the Appendix.

**Step 2.** We shall complete the proof by Moser's iterations. Set, for  $\delta = \delta_2/10$ ,

$$R_l = \mu_\alpha \frac{(2 - 2^{-(l-1)})}{\delta}, \quad l = 1, 2, 3, \dots$$

We choose  $\eta_l$  to be some cutoff function satisfying

$$\begin{aligned}
& \eta_l(x) = 1 \text{ if } |x| \geq R_{l+1}, \quad \eta_l(x) = 0 \text{ if } |x| \leq R_l, \\
& \text{and } \eta_l = \eta_l(|x|) \text{ in the Fermi coordinate system centered at } x_\alpha.
\end{aligned}$$

Since  $\tilde{g}^{ij} \sim \mu_\alpha^2 \delta^{ij}$  in  $\mathcal{B}_{2\mu_\alpha/\delta_2}^+(x_\alpha) \setminus \mathcal{B}_{\mu_\alpha/(4\delta_2)}^+(x_\alpha)$  and  $\eta_l$  is radial in the Fermi coordinate system, we have

$$|\nabla_{\tilde{g}} \eta_l| \leq C2^l, \quad |\operatorname{div}_{\tilde{g}}(\tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta_l^2)| \leq C4^l \tilde{\rho}^{1-2\sigma}, \quad \text{and } \lim_{\rho \rightarrow 0} \tilde{\rho}^{1-2\sigma} \frac{\partial_{\tilde{g}} \eta_l^2}{\partial \tilde{\nu}} = 0.$$

In view of (28), we have

$$\begin{aligned}
& \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}}(w_\alpha^{(k+1)/2} \eta_l)|^2 dv_{\tilde{g}} \\
& \leq C4^l \int_{M \cap (\mathcal{B}_{R_{l+1}}^+(x_\alpha) \setminus \mathcal{B}_{R_l}^+(x_\alpha))} \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} dv_{\tilde{g}} + \frac{C(k+1)^2}{k} \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{q-1+k} ds_{\tilde{g}}.
\end{aligned} \tag{30}$$

Set  $r_0 = s_0/(q-2)$ , where  $s_0$  is given in the step 1. It follows Hölder inequality and (26) that

$$\begin{aligned} \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{q-1+k} ds_{\tilde{g}} &= \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{q-2} w_\alpha^{k+1} ds_{\tilde{g}} \\ &\leq C \left( \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)} ds_{\tilde{g}} \right)^{(r_0-1)/r_0}. \end{aligned} \quad (31)$$

Computing as (29), we see that

$$\begin{aligned} &\int_{M \cap (\mathcal{B}_{R_{l+1}}^+(x_\alpha) \setminus \mathcal{B}_{R_l}^+(x_\alpha))} \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} dv_{\tilde{g}} \\ &\leq C^{k+1} \int_{2^{-2-(l-1)} \leq \delta |z| \leq 2^{-2-l}} \rho_\alpha(z)^{1-2\sigma} v_\alpha(z)^{k+1} dv_{g_\alpha} \\ &\leq C^{k+1} \delta^{-1} 2^{-l} \max_{\mathcal{B}_{2/\delta}^+} v_\alpha^{k+1}, \end{aligned}$$

and

$$\begin{aligned} &\left( \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)} ds_{\tilde{g}} \right)^{(r_0-1)/r_0} \\ &\geq C^{-(k+1)} \left( \int_{1 \leq \delta |z'| \leq 2} \rho_\alpha(z', 0)^{1-2\sigma} v_\alpha(z)^{(k+1)r_0/(r_0-1)} ds_{g_\alpha} \right)^{(r_0-1)/r_0} \\ &\geq C^{-(k+1)} \min_{\partial' \mathcal{B}_{2/\delta}^+} v_\alpha^{k+1}. \end{aligned}$$

Hence, it follows from (19) that

$$\begin{aligned} &\left( \int_{M \cap (\mathcal{B}_{R_{l+1}}^+(x_\alpha) \setminus \mathcal{B}_{R_l}^+(x_\alpha))} \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} dv_{\tilde{g}} \right)^{1/(k+1)} \\ &\leq C \left( \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)} ds_{\tilde{g}} \right)^{(r_0-1)/r_0(k+1)} \end{aligned} \quad (32)$$

It follows from Theorem A.1, (30), (31) and (32) that

$$\begin{aligned} &\left( \int_{\partial M \setminus B_{R_{l+1}}(x_\alpha)} w_\alpha^{(k+1)q/2} ds_{\tilde{g}} \right)^{2/(k+1)q} \\ &\leq \left( C 4^l + \frac{C(k+1)^2}{k} \right)^{1/(k+1)} \left( \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)} ds_{\tilde{g}} \right)^{(r_0-1)/r_0(k+1)}. \end{aligned} \quad (33)$$

Set  $\chi := \frac{r_0-1}{r_0} \cdot \frac{q}{2} = 1 + \frac{(s_0-q)(q-2)}{2s_0} > 1$ ,  $q_0 = \frac{2r_0}{r_0-1}$ ,  $q_l = q_{l-1} \cdot \chi = \chi^{l-1}q$  and  $p_l = q_l(r_0 - 1)/r_0 = 2\chi^l$  where  $l \geq 1$ . Taking  $k = p_l - 1$  in (33), we obtain

$$\|w_\alpha\|_{L^{q_{l+1}}(\partial M \setminus B_{R_{l+1}})} \leq \left(C4^l + \frac{Cp_l^2}{p_l - 1}\right)^{1/p_l} \|w_\alpha\|_{L^{q_l}(\partial M \setminus B_{R_l})}.$$

Therefore,

$$\begin{aligned} \|w_\alpha\|_{L^{q_{l+1}}(\partial M \setminus B_{R_{l+1}})} &\leq \|w_\alpha\|_{L^{q_1}(\partial M \setminus B_{R_1})} \prod_{l=1}^{\infty} \left(C4^l + \frac{Cp_l^2}{p_l - 1}\right)^{1/p_l} \\ &\leq \|w_\alpha\|_{L^{p_1}(\partial M \setminus B_{R_1})} \prod_{l=1}^{\infty} C^{1/(2\chi^l)} (4 + \chi)^{l/(2\chi^l)} \\ &\leq C \|w_\alpha\|_{L^{p_1}(\partial M \setminus B_{R_1})}. \end{aligned}$$

Sending  $l$  to  $\infty$ , we have

$$\|w_\alpha\|_{L^\infty(\partial M \setminus B_{2\mu_\alpha/\delta}(x_\alpha))} \leq C. \quad (34)$$

By the choice of  $G_\alpha$ ,  $\varphi_\alpha(x) \geq C^{-1}\mu_\alpha^{-(n-2\sigma)/2}$  for  $x \in B_{2\mu_\alpha/\delta}(x_\alpha)$ . Hence, for  $x \in B_{2\mu_\alpha/\delta}(x_\alpha)$ ,

$$w_\alpha(x) = \frac{u_\alpha(x)}{\varphi_\alpha(x)} \leq C\mu_\alpha^{(n-2\sigma)/2}u_\alpha(x) \leq C. \quad (35)$$

In view of (34) and (35), we completed the proof of the proposition.  $\square$

**Corollary 3.2.** *There exists a positive constant  $C$  depending only on  $M, g, n, \rho, \sigma$  such that*

$$u_\alpha(x) \leq Cu_\alpha(x_\alpha)^{-1} \text{dist}_{\partial M, g}(x, x_\alpha)^{2\sigma-n}, \quad \text{for all } x \in \partial M.$$

*Proof.* It follows immediately from Proposition 3.3.  $\square$

## 4 Proofs of the main theorems

Let  $u_\alpha$  and  $x_\alpha$  be as in Section 3. We will still use Fermi coordinates  $x = (x_1, \dots, x_{n+1})$  centered at  $x_\alpha$ . In this coordinate system,

$$\sum_{1 \leq i, j \leq n+1} g_{ij}(x) dx_i dx_j = dx_{n+1}^2 + \sum_{1 \leq i, j \leq n} g_{ij}(x) dx_i dx_j, \quad \text{for } |x| \leq \delta_0,$$

where  $\delta_0 > 0$  is independent of  $\alpha$ . Then we have

$$\begin{cases} \text{div}_g \left( \rho(x)^{1-2\sigma} \nabla_g u_\alpha(x) \right) = 0, & \text{in } \mathcal{B}_{\delta_0}^+, \\ - \lim_{x_{n+1} \rightarrow 0^+} \rho(x)^{1-2\sigma} \frac{\partial u_\alpha}{\partial x_{n+1}} = \xi_\alpha u_\alpha^{q-1}(x', 0) - \alpha u_\alpha(x', 0), & \text{on } \partial' \mathcal{B}_{\delta_0}^+. \end{cases} \quad (36)$$

**Proposition 4.1.** *There exists a positive constant  $C$  independent of  $\alpha$  such that*

$$u_\alpha(x) \leq C u_\alpha(0)^{-1} |x|^{2\sigma-n}, \quad \mathcal{B}_{10\alpha^{-1/2\sigma}}^+(0).$$

*Proof.* By Corollary 3.2,

$$u_\alpha(x', 0) \leq C u_\alpha(0)^{-1} |x'|^{2\sigma-n}, \quad |x'| \leq \delta_0. \quad (37)$$

Let  $r := |\bar{x}| < 10\alpha^{-1/2\sigma}$ ,  $\phi_\alpha(x) = r^{\frac{n-2\sigma}{2}} u_\alpha(rx)$ . Then  $\phi_\alpha$  satisfies

$$\begin{cases} \operatorname{div}_{\hat{g}} \left( \hat{\rho}(x)^{1-2\sigma} \nabla_{\hat{g}} \phi_\alpha(x) \right) = 0, & \text{in } \mathcal{B}_{\delta_0/r}^+, \\ - \lim_{x_{n+1} \rightarrow 0^+} \hat{\rho}(x)^{1-2\sigma} \frac{\partial \phi_\alpha}{\partial x_{n+1}} = \xi_\alpha \phi_\alpha^{q-1}(x', 0) - \alpha r^{2\sigma} \phi_\alpha(x', 0), & \text{on } \partial' \mathcal{B}_{\delta_0/r}^+, \end{cases} \quad (38)$$

where  $\hat{\rho}(x) = \rho(rx)/r$ ,  $\hat{g}(x) = g_{ij}(rx) dx_i dx_j$ . Since  $x_\alpha = 0$  is a maximum point of  $u_\alpha$ , it follows from (37) that

$$\phi_\alpha(x', 0) = r^{\frac{n-2\sigma}{2}} u_\alpha(rx', 0) \leq C r^{\frac{n-2\sigma}{2}} (r|x'|)^{-\frac{n-2\sigma}{2}} \leq C, \quad \frac{1}{2} < |x'| < 2. \quad (39)$$

Applying the Harnack inequality in [8] or [43] and standard Harnack inequality for uniformly elliptic equations to  $\phi_\alpha$  in  $\{x : \frac{1}{2} < |x| < 2, x_{n+1} > 0\}$ , we conclude that

$$\max_{\mathcal{B}_{3/2}^+ \setminus \mathcal{B}_{3/4}^+} \phi_\alpha \leq C \min_{\mathcal{B}_{3/2}^+ \setminus \mathcal{B}_{3/4}^+} \phi_\alpha.$$

Hence, by (37)

$$u_\alpha(\bar{x}) \leq C u(\tilde{x}', 0) \leq C u_\alpha(0)^{-1} |\bar{x}|^{2\sigma-n},$$

where  $|\tilde{x}'| = |\bar{x}|$ . By the arbitrary choice of  $\bar{x}$ , the proposition follows immediately.  $\square$

Let  $\mu_\alpha = u_\alpha(0)^{-\frac{2}{n-2\sigma}}$ ,  $R_\alpha = (\alpha^{1/2\sigma} \mu_\alpha)^{-1}$ ,  $g_\alpha = g_{ij}(\mu_\alpha x) dx_i dx_j$  and  $\rho_\alpha(x) = \frac{\rho(\mu_\alpha x)}{\mu_\alpha}$  in  $\mathcal{B}_{10R_\alpha}^+$ . Set  $v_\alpha(x) = \mu_\alpha^{\frac{n-2\sigma}{2}} u_\alpha(\mu_\alpha x)$  for  $x \in \mathcal{B}_{10R_\alpha}^+$ . It follows that

$$\begin{cases} \operatorname{div}_{g_\alpha} \left( \rho_\alpha^{1-2\sigma} \nabla_{g_\alpha} v_\alpha \right) = 0, & \text{in } \mathcal{B}_{10R_\alpha}^+, \\ \lim_{x_{n+1} \rightarrow 0} \rho_\alpha^{1-2\sigma} \frac{\partial_{g_\alpha} v_\alpha}{\partial \nu} = \xi_\alpha v_\alpha^{q-1} - \alpha \mu_\alpha^{2\sigma} v_\alpha, & \text{on } \partial' \mathcal{B}_{10R_\alpha}^+ = B_{10R_\alpha}, \\ v_\alpha(0) = 1, \quad 0 < v_\alpha \leq 1. \end{cases} \quad (40)$$

By Proposition 4.1,

$$v_\alpha(x) \leq \frac{C}{1 + |x|^{n-2\sigma}}, \quad x \in \overline{\mathcal{B}}_{10R_\alpha}^+. \quad (41)$$

**Proposition 4.2.** For all  $\alpha \geq 1$ ,  $x \in \mathcal{B}_{R_\alpha}^+(0)$ , we have

$$\begin{aligned} |\nabla_{x'} v_\alpha(x', x_{n+1})| &\leq \frac{C}{1 + |x|^{n+1-2\sigma}}, \\ |\nabla_{x'}^2 v_\alpha(x', x_{n+1})| &\leq \frac{C}{1 + |x|^{n+2-2\sigma}}, \\ |\partial_{n+1} v_\alpha(x', x_{n+1})| &\leq \frac{C x_{n+1}^{2\sigma-1}}{1 + |x|^n}. \end{aligned}$$

*Proof.* Given Theorem A.3 and Proposition A.1, the proofs follow from (41) and standard rescaling arguments (see, e.g., Proposition 3.1 of [32]).  $\square$

*Proof of Theorem 1.1.* We complete the proof of Theorem 1.1 by checking balance via a Pohozaev type inequality.

It follows from direct computations that

$$\begin{aligned} &2\operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha)(\nabla v_\alpha \cdot x) \\ &= \operatorname{div}(2x_{n+1}^{1-2\sigma} (\nabla v_\alpha \cdot x) \nabla v_\alpha - x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 x) + (n - 2\sigma) x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2. \end{aligned} \quad (42)$$

Integrating both sides of (42) over  $\mathcal{B}_{R_\alpha}^+$ , we have

$$\begin{aligned} &\int_{\mathcal{B}_{R_\alpha}^+} \operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha)(\nabla v_\alpha \cdot x) \, dx - \frac{n - 2\sigma}{2} \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 \, dx \\ &= \frac{1}{2} \int_{\mathcal{B}_{R_\alpha}^+} \operatorname{div}(2x_{n+1}^{1-2\sigma} (\nabla v_\alpha \cdot x) \nabla v_\alpha - x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 x) \, dx. \end{aligned} \quad (43)$$

Integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2} \int_{\mathcal{B}_{R_\alpha}^+} \operatorname{div}(2x_{n+1}^{1-2\sigma} (\nabla v_\alpha \cdot x) \nabla v_\alpha - x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 x) \, dx \\ &= - \int_{\partial' \mathcal{B}_{R_\alpha}^+} \left( \sum_{i=1}^n x_i \frac{\partial v_\alpha}{\partial x_i} \right) \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} \, dx' + \int_{\partial'' \mathcal{B}_{R_\alpha}^+} |x| x_{n+1}^{1-2\sigma} \left( \left( \frac{\partial v_\alpha}{\partial \nu} \right)^2 - \frac{1}{2} |\nabla v_\alpha|^2 \right) \, dS \\ &= - \int_{\partial' \mathcal{B}_{R_\alpha}^+} \left( \sum_{i=1}^n x_i \frac{\partial v_\alpha}{\partial x_i} \right) \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} \, dx' + \int_{\partial'' \mathcal{B}_{R_\alpha}^+} \frac{|x|}{2} x_{n+1}^{1-2\sigma} \left( \left( \frac{\partial v_\alpha}{\partial \nu} \right)^2 - |\partial_{\tan} v_\alpha|^2 \right) \, dS, \end{aligned}$$

where  $\frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} := \lim_{x_{n+1} \rightarrow 0^+} x_{n+1}^{1-2\sigma} \frac{\partial v_\alpha}{\partial x_{n+1}}$  and  $\partial_{\tan}$  denotes the tangential differentiation on  $\partial'' \mathcal{B}_{R_\alpha}^+$ .

On the other hand,

$$\begin{aligned} \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 \, dx &= - \int_{\mathcal{B}_{R_\alpha}^+} \operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) v_\alpha \, dx \\ &\quad - \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} \, dx' + \int_{\partial'' \mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} v_\alpha \frac{\partial v_\alpha}{\partial \nu} \, dS. \end{aligned}$$

In summary, we obtain

$$\begin{aligned} & \int_{\mathcal{B}_{R_\alpha}^+} \operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) (\nabla v_\alpha \cdot x) \, dx + \frac{n-2\sigma}{2} \int_{\mathcal{B}_{R_\alpha}^+} \operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) v_\alpha \, dx \\ &= B'(R_\alpha, v_\alpha) + B''(R_\alpha, v_\alpha), \end{aligned} \quad (44)$$

where

$$\begin{aligned} B'(R_\alpha, v_\alpha) &= -\frac{1}{2} \int_{\partial' \mathcal{B}_{R_\alpha}^+} 2 \left( \sum_{i=1}^n x_i \frac{\partial v_\alpha}{\partial x_i} \right) \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} + (n-2\sigma) v_\alpha \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} \, dx', \\ B''(R_\alpha, v_\alpha) &= \frac{1}{2} \int_{\partial'' \mathcal{B}_{R_\alpha}^+} |x| x_{n+1}^{1-2\sigma} \left( \left( \frac{\partial v_\alpha}{\partial \nu} \right)^2 - |\partial_{\tan} v_\alpha|^2 \right) + (n-2\sigma) x_{n+1}^{1-2\sigma} v_\alpha \frac{\partial v_\alpha}{\partial \nu} \, dS. \end{aligned}$$

Note that

$$\begin{aligned} & \operatorname{div}_{g_\alpha}(\rho_\alpha^{1-2\sigma} \nabla_{g_\alpha} v_\alpha) \\ &= g_\alpha^{ij} \frac{\partial v_\alpha}{\partial x_i} \frac{\partial \rho_\alpha^{1-2\sigma}}{\partial x_j} + \rho_\alpha^{1-2\sigma} g_\alpha^{ij} \left( \frac{\partial^2 v_\alpha}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial v_\alpha}{\partial x_k} \right) \\ &= \operatorname{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) + \sum_{1 \leq i, j \leq n} g_\alpha^{ij} \frac{\partial v_\alpha}{\partial x_i} \frac{\partial \rho_\alpha^{1-2\sigma}}{\partial x_j} + \left( \frac{\partial \rho_\alpha^{1-2\sigma}}{\partial x_{n+1}} - \frac{\partial x_{n+1}^{1-2\sigma}}{\partial x_{n+1}} \right) \frac{\partial v_\alpha}{\partial x_{n+1}} \\ &\quad + \rho_\alpha^{1-2\sigma} (g_\alpha^{ij} - \delta^{ij}) \frac{\partial^2 v_\alpha}{\partial x_i \partial x_j} + (\rho_\alpha^{1-2\sigma} - x_{n+1}^{1-2\sigma}) \Delta v_\alpha - \rho_\alpha^{1-2\sigma} g_\alpha^{ij} \Gamma_{ij}^k \frac{\partial v_\alpha}{\partial x_k}, \end{aligned} \quad (45)$$

where  $\Gamma_{ij}^k$  is the Christoffel symbol of  $g_\alpha$ . It is easy to see that

$$|h_\alpha^{ij}(x) - \delta^{ij}| \leq C \mu_\alpha |x|, \quad (46a)$$

$$|\Gamma_{ij}^k| \leq C \mu_\alpha, \quad (46b)$$

$$|\rho_\alpha(x)^{1-2\sigma} - x_{n+1}^{1-2\sigma}| \leq C \mu_\alpha x_{n+1}^{2-2\sigma}, \quad (46c)$$

$$\left| \frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_i} \right| \leq C \mu_\alpha x_{n+1}^{1-2\sigma} \quad \text{for } i < n+1, \quad (46d)$$

$$\left| \frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_{n+1}} - \frac{\partial x_{n+1}^{1-2\sigma}}{\partial x_{n+1}} \right| \leq C \mu_\alpha x_{n+1}^{1-2\sigma}. \quad (46e)$$

Indeed,

$$\begin{aligned} |\rho_\alpha(x)^{1-2\sigma} - x_{n+1}^{1-2\sigma}| &= x_{n+1}^{1-2\sigma} \left| \left( \frac{\rho(\mu_\alpha x)}{\mu_\alpha x_{n+1}} \right)^{1-2\sigma} - 1 \right| \\ &= x_{n+1}^{1-2\sigma} \left| \left( \frac{\mu_\alpha x_{n+1} + O(\mu_\alpha x_{n+1})^2}{\mu_\alpha x_{n+1}} \right)^{1-2\sigma} - 1 \right| \\ &\leq C \mu_\alpha x_{n+1}^{2-2\sigma}, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_i} &= (1-2\sigma)\rho_\alpha(x)^{-2\sigma} \left( \frac{\partial \rho_\alpha(x)}{\partial x_i} - \frac{\partial \rho_\alpha(x',0)}{\partial x_i} \right) \\ &= O(1)\mu_\alpha \rho_\alpha^{1-2\sigma} \\ &\leq C\mu_\alpha x_{n+1}^{1-2\sigma}.\end{aligned}$$

It follows from (40), (44), (45) and (46a)-(46e) that

$$\begin{aligned}B'(R_\alpha, v_\alpha) + B''(R_\alpha, v_\alpha) \\ \leq C\mu_\alpha \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|)(|\nabla v_\alpha| + |x||\nabla_{x'}^2 v_\alpha| + x_{n+1}|\Delta v_\alpha|) dx.\end{aligned}\quad (47)$$

Since  $\lim_{x_{n+1} \rightarrow 0} \rho_\alpha^{1-2\sigma} \frac{\partial g_\alpha v_\alpha}{\partial \nu} = -\frac{\partial v_\alpha}{\partial x_{n+1}^\sigma}$  on  $\partial' \mathcal{B}_{R_\alpha}^+$ ,

$$\begin{aligned}B'(R_\alpha, v_\alpha) &= \int_{\partial' \mathcal{B}_{R_\alpha}^+} \left( \sum_{i=1}^n x_i \frac{\partial v_\alpha}{\partial x_i} \right) (\xi_\alpha v_\alpha^{q-1} - \alpha \mu_\alpha^{2\sigma} v_\alpha) + \frac{(n-2\sigma)}{2} (\xi_\alpha v_\alpha^q - \alpha \mu_\alpha^{2\sigma} v_\alpha^2) dx' \\ &= \sigma \alpha \mu_\alpha^{2\sigma} \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 dx' + \int_{\partial B_{R_\alpha}} \left( \frac{\xi_\alpha}{q} v_\alpha^q - \frac{\alpha \mu_\alpha^{2\sigma}}{2} v_\alpha^2 \right) R_\alpha dS,\end{aligned}$$

where integrations by parts were used in the second equality. Clearly,

$$B''(R_\alpha, v_\alpha) = O \left( \int_{\partial'' \mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (|x||\nabla v_\alpha|^2 + v_\alpha |\nabla v_\alpha|) dS \right).$$

Therefore, we obtain

$$\begin{aligned}\alpha \mu_\alpha^{2\sigma} \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 dx' \\ \leq C\mu_\alpha \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|)(|\nabla v_\alpha| + |x||\nabla_{x'}^2 v_\alpha| + x_{n+1}|\Delta v_\alpha|) dx \\ + C \int_{\partial'' \mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (|x||\nabla v_\alpha|^2 + v_\alpha |\nabla v_\alpha|) dS + C \int_{\partial B_{R_\alpha}} \alpha \mu_\alpha^{2\sigma} v_\alpha^2 R_\alpha dS.\end{aligned}\quad (48)$$

Since  $\operatorname{div}_{g_\alpha}(\rho_\alpha^{1-2\sigma} \nabla_{g_\alpha} v_\alpha) = 0$  and  $g_\alpha^{i,n+1} = 0$  for  $i < n+1$ ,

$$|\partial_{n+1}^2 v_\alpha(x', x_{n+1})| \leq C(\mu_\alpha |\nabla v_\alpha| + |\partial_{x+1} v_\alpha| x_{n+1}^{-1} + |\nabla_{x'}^2 v_\alpha|). \quad (49)$$

It follows from (48), (49) and Proposition 4.2 that

$$\begin{aligned}
& \alpha \mu_\alpha^{2\sigma} \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 \, dx' \\
& \leq C \mu_\alpha \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|)(|\nabla v_\alpha| + |x| |\nabla_{x'}^2 v_\alpha|) \, dx \\
& \quad + C \int_{\partial'' \mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} \left( \frac{1}{R_\alpha^{2n+1-4\sigma}} + \frac{x_{n+1}^{2\sigma-1}}{R_\alpha^{2n-2\sigma}} + \frac{x_{n+1}^{4\sigma-2}}{R_\alpha^{2n-1}} \right) \, dS + C \frac{\alpha \mu_\alpha^{2\sigma}}{R_\alpha^{n-4\sigma}} \\
& \leq C \mu_\alpha \int_{\mathcal{B}_{R_\alpha}^+} \left( \frac{x_{n+1}^{1-2\sigma}}{(1+|x|)^{2n+1-4\sigma}} + \frac{1}{(1+|x|)^{2n-2\sigma}} \right) \, dx \\
& \quad + C R_\alpha^{2\sigma-n} \int_{\partial'' \mathcal{B}_1} (y_{n+1}^{1-2\sigma} + 1 + y_{n+1}^{2\sigma-1}) \, dS + C \frac{\alpha \mu_\alpha^{2\sigma}}{R_\alpha^{n-4\sigma}} \\
& \leq \begin{cases} C \mu_\alpha \ln R_\alpha + C(\alpha \mu_\alpha^{2\sigma})^{\frac{n-2\sigma}{2\sigma}} + C \alpha \mu_\alpha^{2\sigma} R_\alpha^{4\sigma-n}, & n = 2\sigma + 1 \\ C \mu_\alpha + C(\alpha \mu_\alpha^{2\sigma})^{\frac{n-2\sigma}{2\sigma}} + C \alpha \mu_\alpha^{2\sigma} R_\alpha^{4\sigma-n}, & n > 2\sigma + 1. \end{cases}
\end{aligned}$$

For  $\sigma = 1/2$  and  $n = 2$ , Theorem 1.1 was proved in [32]. Hence, we may assume that  $n > 2\sigma + 1$ . Since  $\sigma \in (0, 1/2]$ ,  $n > 2\sigma + 1 \geq 4\sigma$ . Therefore,

$$0 < \frac{1}{C} \leq \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 \, dx' \rightarrow 0, \quad \text{as } \alpha \rightarrow \infty$$

which is a contradiction.  $\square$

*Proof of Theorem 1.2.* Since  $\partial M$  is totally geodesic, Lemma 3.2 implies that

$$|h_\alpha^{ij}(x) - \delta^{ij}| \leq C \mu_\alpha^2 |x|^2, \quad (50a)$$

$$|\Gamma_{ij}^k| \leq C \mu_\alpha^2 |x|. \quad (50b)$$

Since  $\rho = d(x) + O(d(x)^3)$ , it follows that

$$|\rho_\alpha(x)^{1-2\sigma} - x_{n+1}^{1-2\sigma}| \leq C \mu_\alpha^2 x_{n+1}^{3-2\sigma}, \quad (51a)$$

$$\left| \frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_i} \right| \leq C \mu_\alpha^2 x_{n+1}^{2-2\sigma}, \quad i < n+1, \quad (51b)$$

$$\left| \frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_{n+1}} - \frac{\partial x_{n+1}^{1-2\sigma}}{\partial x_{n+1}} \right| \leq C \mu_\alpha^2 x_{n+1}^{2-2\sigma}. \quad (51c)$$

Similar to (48), we have

$$\begin{aligned}
& \alpha \mu_\alpha^{2\sigma} \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 dx' \\
& \leq C \mu_\alpha^2 \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|) (|x| |\nabla v_\alpha| + |x|^2 |\nabla_{x'}^2 v_\alpha| + x_{n+1}^2 |\Delta v_\alpha|) dx \\
& \quad + C \int_{\partial'' \mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (|x| |\nabla v_\alpha|^2 + v_\alpha |\nabla v_\alpha|) dS + C \int_{\partial B_{R_\alpha}} \alpha \mu_\alpha^{2\sigma} v_\alpha^2 R_\alpha dS.
\end{aligned} \tag{52}$$

It follows from (49), (52) and Proposition 4.2 that

$$\begin{aligned}
& \alpha \mu_\alpha^{2\sigma} \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 dx' \\
& \leq C \mu_\alpha^2 \int_{\mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|) (|x| |\nabla v_\alpha| + |x|^2 |\nabla_{x'}^2 v_\alpha|) dx \\
& \quad + C \int_{\partial'' \mathcal{B}_{R_\alpha}^+} x_{n+1}^{1-2\sigma} (|x| |\nabla v_\alpha|^2 + v_\alpha |\nabla v_\alpha|) dS + C \int_{\partial B_{R_\alpha}} \alpha \mu_\alpha^{2\sigma} v_\alpha^2 R_\alpha dS \\
& \leq C \mu_\alpha^2 \int_{\mathcal{B}_{R_\alpha}^+} \frac{x_{n+1}^{1-2\sigma}}{(1+|x|)^{2n-4\sigma}} dx + C (\alpha \mu_\alpha^{2\sigma})^{\frac{n-2\sigma}{2\sigma}} + C \alpha \mu_\alpha^{2\sigma} R_\alpha^{4\sigma-n} \\
& \leq C \mu_\alpha^2 + C (\alpha \mu_\alpha^{2\sigma})^{\frac{n-2\sigma}{2\sigma}} + C \alpha \mu_\alpha^{2\sigma} R_\alpha^{4\sigma-n},
\end{aligned}$$

provided  $n > 2 + 2\sigma$  (i.e.,  $n \geq 4$ ). Therefore,

$$0 < \frac{1}{C} \leq \int_{\partial' \mathcal{B}_{R_\alpha}^+} v_\alpha^2 dx' \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty,$$

which is a contradiction.  $\square$

## A Appendix

### A.1 A trace inequality

Let  $(M, g)$  be a smooth, compact Riemannian manifold of dimension  $n+1$  ( $n \geq 2$ ) with boundary.

**Lemma A.1.** *For  $n \geq 2$ , there exists some positive constant  $C = C(n, \sigma)$  such that for all  $u \in H^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_1^+)$ ,  $u \equiv 0$  in an open neighborhood of  $x = 0$ , we have*

$$\left( \int_{\partial' \mathcal{B}_1^+} \frac{|u(x', 0)|^q}{|x'|^{2n}} dx' \right)^{2/q} \leq C \int_{\mathcal{B}_1^+} \frac{x_{n+1}^{1-2\sigma} |\nabla u|^2}{|x|^{2n-4\sigma}} dx.$$

*Proof.* By the assumption of  $u$ , there exists a positive constant  $\mu = \mu(u) > 0$  such that  $u \equiv 0$  for  $|x| < \mu$  with  $x_{n+1} > 0$ . Consider

$$v(y) = u\left(\frac{y}{|y|^2}\right), \quad |y| > 1, y_{n+1} > 0.$$

It is easy to see that

$$v(y) \equiv 0, \quad \text{for all } |y| > 1/\mu, y_{n+1} > 0,$$

and for some  $C(n) > 0$ ,

$$\int_{\partial' \mathcal{B}_1^+} \frac{|u(x', 0)|^q}{|x'|^{2n}} dx' = C(n) \int_{|y'| \geq 1} |v(y', 0)|^q dy',$$

and

$$\int_{\mathcal{B}_1^+} \frac{x_{n+1}^{1-2\sigma} |\nabla u|^2}{|x|^{2n-4\sigma}} dx = C(n) \int_{|y| \geq 1, y_{n+1} > 0} y_{n+1}^{1-2\sigma} |\nabla v(y)|^2 dy.$$

By some appropriate extension of  $v$  to  $|y| < 1$ , it follows from (3) that

$$\int_{|y'| \geq 1} |v(y', 0)|^q dy' \leq C(n, \sigma) \int_{|y| \geq 1, y_{n+1} > 0} y_{n+1}^{1-2\sigma} |\nabla v(y)|^2 dy.$$

The proof is completed.  $\square$

**Lemma A.2.** For  $\delta > 0$ , there exists  $C = C(M, g, n, \sigma, \delta, \rho) > 0$  such that for all  $x_0 \in \partial M$ ,  $u \in H^1(\rho^{1-2\sigma}, M \setminus \mathcal{B}_{\delta/2}(x_0))$ , we have

$$\begin{aligned} & \left( \int_{\partial M \setminus B_\delta(x_0)} |u(x)|^q \right)^{2/q} + \int_{M \setminus \mathcal{B}_\delta^+(x_0)} \rho^{1-2\sigma} |u(x)|^2 \\ & \leq C \left\{ \int_{M \setminus \mathcal{B}_{\delta/2}^+(x_0)} \rho^{1-2\sigma} |\nabla_g u|^2 + \int_{\partial M \cap (B_\delta(x_0) \setminus \overline{B}_{\delta/2}(x_0))} |u(x)|^2 \right\}. \end{aligned} \quad (53)$$

*Proof.* We prove (53) by contradiction. Suppose the contrary of (53) that for some  $\delta > 0$ , there exists a sequence of points  $\{x_i\} \in \partial M$ ,  $\{u_i\} \in H^1(\rho^{1-2\sigma}, M \setminus \mathcal{B}_{\delta/2}^+(x_i))$  satisfying

$$\left( \int_{\partial M \setminus B_\delta(x_i)} |u_i(x)|^q \right)^{2/q} + \int_{M \setminus \mathcal{B}_\delta^+(x_i)} \rho^{1-2\sigma} |u_i(x)|^2 = 1, \quad (54)$$

but

$$\int_{M \setminus \mathcal{B}_{\delta/2}^+(x_i)} \rho^{1-2\sigma} |\nabla_g u_i|^2 + \int_{\partial M \cap (B_\delta(x_i) \setminus \overline{B}_{\delta/2}(x_i))} |u_i(x)|^2 \leq \frac{1}{i}. \quad (55)$$

After passing to some subsequence,  $\{u_i\}$  converges weakly to  $u$  in  $H^1(\rho^{1-2\sigma}, M \setminus \mathcal{B}_\delta^+(x_i))$ . By (55),  $u \equiv 0$ . It follows from a compact Sobolev embedding in Proposition A.2 that

$$\int_{M \setminus \mathcal{B}_\delta^+(x_i)} \rho^{1-2\sigma} |u_i(x)|^2 \rightarrow 0.$$

By a trace embedding in Proposition 2.3, we also conclude that

$$\left( \int_{\partial M \setminus B_\delta(x_i)} |u(x)|^q \right)^{2/q} \rightarrow 0.$$

Therefore, we reach a contradiction to (54).  $\square$

**Theorem A.1.** *There exists some constant  $C = C(M, g, \rho, n, \sigma)$  such that for all  $x_0 \in \partial M$ ,  $\mu > 0$ ,  $u \in H^1(\rho^{1-2\sigma}, M)$ ,  $u \equiv 0$  in  $\{x \in M : \text{dist}(x, x_0) < \mu\}$ , we have*

$$\left( \int_{\partial M} \frac{|u(x)|^q}{\text{dist}(x, x_0)^{2n}} ds_g \right)^{2/q} \leq C \int_M \frac{\rho^{1-2\sigma} |\nabla_g u|^2}{\text{dist}(x, x_0)^{2n-4\sigma}} dv_g.$$

*Proof.* The theorem follows clearly from Lemma A.1 and Lemma A.2.  $\square$

## A.2 Regularity results for degenerate elliptic equations

Suppose that  $a^{ij}(x)$ ,  $1 \leq i, j \leq n+1$ , is a smooth positive definite matrix-valued in  $\mathcal{B}_2^+$  and there exists a positive constant  $\Lambda \geq 1$  such that

$$\frac{1}{\Lambda} |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{n+1}$$

Suppose also that

$$a^{i, n+1} = a^{n+1, i} = 0 \text{ for } i < n+1.$$

Consider

$$\begin{cases} \frac{\partial}{\partial x_i} (x_{n+1}^{1-2\sigma} a^{ij}(x) \frac{\partial}{\partial x_j} u(x)) = 0, & \text{in } \mathcal{B}_2^+, \\ - \lim_{x_{n+1} \rightarrow 0^+} x_{n+1}^{1-2\sigma} a^{n+1, n+1} \frac{\partial u(x)}{\partial x_{n+1}} = b(x')u + f(x'), & \text{on } \partial' \mathcal{B}_2^+. \end{cases} \quad (56)$$

We say  $u \in H^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)$  is a weak solution of (56) if

$$\int_{\mathcal{B}_2^+} x_{n+1}^{1-2\sigma} a^{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} = \int_{\partial' \mathcal{B}_2^+} b(x')u(x', 0) \varphi(x', 0) + f(x') \varphi(x', 0)$$

for all  $\varphi \in C_c^\infty(\mathcal{B}_2^+ \cup \partial' \mathcal{B}_2^+)$ .

**Theorem A.2.** Suppose that  $b, f \in L^p(B_2)$  for some  $p > \frac{n}{2\sigma}$ . Let  $u \in H^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)$  be a weak solution of (56). Then there exist constants  $\gamma \in (0, 1)$ ,  $C > 0$  depending only on  $n, \sigma, \Lambda, p, \|b\|_{L^p(B_2)}$  such that  $u \in C^\gamma(\mathcal{B}_1^+)$  and

$$\|u\|_{C^\gamma(\mathcal{B}_1^+)} \leq C(\|u\|_{L^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)} + \|f\|_{L^p(B_2)}).$$

*Proof.* It follows from a modification of the proof of Proposition 2.4 in [26], which uses standard Moser iteration techniques.  $\square$

**Theorem A.3.** Suppose that  $b, f \in C^\beta(B_2)$  for some  $0 < \beta \notin \mathbb{N}$ . Let  $u \in H^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)$  be a weak solution of (56). Suppose that  $2\sigma + \beta$  is not an integer. Then  $x_{n+1}^{1-2\sigma} \frac{\partial u(x)}{\partial x_{n+1}} \in C(\overline{\mathcal{B}_1^+})$ , and  $u(\cdot, 0) \in C^{2\sigma+\beta}(B_1)$ . Moreover,

$$\left| x_{n+1}^{1-2\sigma} \frac{\partial u(x)}{\partial x_{n+1}} \right|_{C(\overline{\mathcal{B}_1^+})} + \|u(\cdot, 0)\|_{C^{2\sigma+\beta}(B_1)} \leq C(\|u\|_{L^2(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)} + \|f\|_{C^\beta(B_2)}),$$

where  $C > 0$  depending only on  $n, \sigma, \Lambda, \beta, \|b\|_{C^\beta(B_2)}$ .

*Proof.* It follows from modifications of the proofs of Theorem 2.3 and Lemma 2.3 in [26].  $\square$

**Proposition A.1.** Let  $b, f \in C^k(B_2)$ ,  $u \in H^1(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)$  be a weak solution of (56), where  $k$  is a positive integer. Then we have

$$\sum_{j=1}^k \|\nabla_{x'}^j u\|_{L^\infty(\mathcal{B}_1^+)} \leq C(\|u\|_{L^2(x_{n+1}^{1-2\sigma}, \mathcal{B}_2^+)} + \|f\|_{C^k(B_2)}),$$

where  $C > 0$  depending only on  $n, \sigma, \Lambda, \beta, \|b\|_{C^k(B_2)}$ .

*Proof.* It follows from a modification of the proof of Proposition 2.5 in [26].  $\square$

### A.3 Degenerate elliptic equations with conormal boundary conditions involving measures

We start with some Sobolev embeddings. For every  $p \in [1, +\infty)$ , we define  $W^{1,p}(\rho^{1-2\sigma}, M)$  as the closure of  $C^\infty(\overline{M})$  under the norm

$$\|u\|_{W^{1,p}(\rho^{1-2\sigma}, M)} = \left( \int_M \rho^{1-2\sigma} (|u|^p + |\nabla u|^p) dv_g \right)^{\frac{1}{p}},$$

where  $dv_g$  denote the volume form of  $(M, g)$ .  $W^{1,p}(\rho^{1-2\sigma}, M)$  is a Banach space for all  $p \in [1, +\infty)$  (see [30]). The following Proposition follows directly from Theorem 8.8 and Theorem 8.12 in [23].

**Proposition A.2.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^{n+1}$  with Lipschitz boundary  $\partial\Omega$ . Let  $\sigma \in (0, 1)$ ,  $1 \leq p \leq q < \infty$  with  $\frac{1}{n+1} > \frac{1}{p} - \frac{1}{q}$  and  $d(x)$  be the distance from  $x$  to  $\partial\Omega$ .

(i) Suppose that  $2 - 2\sigma \leq p$ . Then  $W^{1,p}(d^{1-2\sigma}, \Omega)$  is compactly embedded in  $L^q(d^{1-2\sigma}, \Omega)$  if

$$\frac{2 - 2\sigma}{p(n + 2 - 2\sigma)} > \frac{1}{p} - \frac{1}{q}.$$

(ii) Suppose that  $2 - 2\sigma > p$ . Then  $W^{1,p}(d^{1-2\sigma}, \Omega)$  is compactly embedded in  $L^q(d^{1-2\sigma}, \Omega)$  if and only if

$$\frac{1}{n + 2 - 2\sigma} > \frac{1}{p} - \frac{1}{q}.$$

**Corollary A.1.** For  $n \geq 2$ , let  $(M, g)$  be an  $n + 1$  dimensional, compact, smooth Riemannian manifold with smooth boundary  $\partial M$ . Let  $\sigma \in (0, 1)$ , and  $\rho$  be a defining function of  $M$  with  $|\nabla_g \rho| = 1$  on  $\partial M$ . Let  $1 \leq p \leq q < \infty$  with  $\frac{1}{n+1} > \frac{1}{p} - \frac{1}{q}$ .

(i) Suppose that  $2 - 2\sigma \leq p$ . Then  $W^{1,p}(\rho^{1-2\sigma}, M)$  is compactly embedded in  $L^q(d^{1-2\sigma}, M)$  if

$$\frac{2 - 2\sigma}{p(n + 2 - 2\sigma)} > \frac{1}{p} - \frac{1}{q}.$$

(ii) Suppose that  $2 - 2\sigma > p$ . Then  $W^{1,p}(\rho^{1-2\sigma}, M)$  is compactly embedded in  $L^q(d^{1-2\sigma}, M)$  if and only if

$$\frac{1}{n + 2 - 2\sigma} > \frac{1}{p} - \frac{1}{q}.$$

*Proof.* It follows from Proposition A.2 and partition of unity.  $\square$

**Proposition A.3.** For  $n \geq 2$ , let  $(M, g)$  be an  $n + 1$  dimensional, compact, smooth Riemannian manifold with smooth boundary  $\partial M$ . Let  $\sigma \in (0, 1)$ ,  $\rho$  be a defining function of  $M$  with  $|\nabla_g \rho| = 1$  on  $\partial M$ , and  $(u)_{M,\rho} = \int_M \rho^{1-2\sigma} u \, dV_g / \int_M \rho^{1-2\sigma} \, dV_g$ . Let  $1 < p < \infty$ . Then there exists a constant  $C$ , depending only on  $M, g, p, n, \sigma$  and  $\rho$ , such that

$$\|u - (u)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)} \leq C \|\nabla_g u\|_{L^p(\rho^{1-2\sigma}, M)} \quad (57)$$

for every function  $u \in W^{1,p}(\rho^{1-2\sigma}, M)$ .

*Proof.* We argue by contradiction. Were the stated estimate false, there would exist for each integer  $k = 1, 2, \dots$  a function  $u_k \in W^{1,p}(\rho^{1-2\sigma}, M)$  satisfying

$$\|u_k - (u_k)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)} > k \|\nabla_g u_k\|_{L^p(\rho^{1-2\sigma}, M)}.$$

For each  $k$ , define

$$v_k := \frac{u - (u)_{M,\rho}}{\|u - (u)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)}}.$$

Then

$$(v_k)_{M,\rho} = 0, \quad \|v_k\|_{L^p(\rho^{1-2\sigma}, M)} = 1, \quad \|\nabla_g v_k\|_{L^p(\rho^{1-2\sigma}, M)} < 1/k.$$

By Corollary A.1, there exists a subsequence of  $\{v_k\}$ , which is still denoted as  $\{v_k\}$ , and a function  $v \in L^p(\rho^{1-2\sigma}, M)$  such that

$$v_k \rightarrow v \text{ in } L^p(\rho^{1-2\sigma}, M), \quad v_k \rightharpoonup v \text{ in } W^{1,p}(\rho^{1-2\sigma}, M).$$

Consequently,

$$(v)_{M,\rho} = 0, \quad \|v\|_{L^p(\rho^{1-2\sigma}, M)} = 1, \quad \|\nabla_g v\|_{L^p(\rho^{1-2\sigma}, M)} \leq \liminf_{k \rightarrow \infty} \|\nabla_g v_k\|_{L^p(\rho^{1-2\sigma}, M)} = 0.$$

We reach a contradiction.  $\square$

**Corollary A.2.** For  $n \geq 2$ , let  $(M, g)$  be an  $n + 1$  dimensional, compact, smooth Riemannian manifold with smooth boundary  $\partial M$ . Let  $\sigma \in (0, 1)$ ,  $\rho$  be a defining function of  $M$  with  $|\nabla_g \rho| = 1$  on  $\partial M$ , and  $(u)_{M,\rho} = \int_M \rho^{1-2\sigma} u \, dV_g / \int_M \rho^{1-2\sigma} \, dV_g$ . Let  $1 < p < \infty$ . Then there exists a constant  $\delta_0$  depending only on  $n, \sigma, p$  such that for any  $1 \leq k \leq 1 + \delta_0$ ,

$$\|u - (u)_{M,\rho}\|_{L^{kp}(\rho^{1-2\sigma}, M)} \leq C \|\nabla_g u\|_{L^p(\rho^{1-2\sigma}, M)} \quad (58)$$

for every function  $u \in W^{1,p}(\rho^{1-2\sigma}, M)$ , where  $C$  is a positive constant depending only on  $M, g, p, n, \sigma$  and  $\rho$ ,

*Proof.* By Corollary A.1, there exists a constant  $\delta_0$  depending only on  $n, \sigma, p$  such that for any  $1 \leq k \leq 1 + \delta_0$ ,

$$\begin{aligned} \|u - (u)_{M,\rho}\|_{L^{kp}(\rho^{1-2\sigma}, M)} &\leq C \|\nabla_g u\|_{L^p(\rho^{1-2\sigma}, M)} + C \|u - (u)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)} \\ &\leq C \|\nabla_g u\|_{L^p(\rho^{1-2\sigma}, M)} \end{aligned}$$

where in the last inequality we have used Proposition A.3.  $\square$

Let  $(M, g)$ ,  $\rho$  be as in Theorem 1.1. For  $\sigma \in (0, 1)$ , we consider

$$\begin{cases} \operatorname{div}_g(\rho^{1-2\sigma} \nabla_g u) = 0, & \text{in } M \\ \lim_{y \rightarrow x \in \partial M} \rho(y)^{1-2\sigma} \frac{\partial_g u}{\partial \nu} = f(x) & \text{on } \partial M. \end{cases} \quad (59)$$

We say  $u \in W^{1,1}(\rho^{1-2\sigma}, M)$  is a weak solution of (59) if

$$\int_M \rho^{1-2\sigma} \langle \nabla_g u, \nabla_g \varphi \rangle \, dv_g = \int_{\partial' M} f \varphi \, ds_g \quad (60)$$

for all  $\varphi \in C^\infty(\overline{M})$ . Define  $\tilde{H}^1 := \{u \in H^1(\rho^{1-2\sigma}, M) : \int_M \rho^{1-2\sigma} u \, dv_g = 0\}$ .

**Lemma A.3.** Let  $f \in H^{-\sigma}(\partial M) := (H^\sigma(\partial M))^*$ , the dual of  $H^{-\sigma}(\partial M)$ , such that  $\langle f, 1 \rangle = 0$ . Then (59) admits a unique weak solution  $u \in \tilde{H}^1$ .

*Proof.* The lemma follows immediately from Proposition A.3 and the Lax-Milgram theorem.  $\square$

**Lemma A.4.** Let  $f \in L^2(\partial M)$  with zero mean value,  $u \in \tilde{H}^1$  be the weak solution of (59). Then for any  $\theta > 1$ ,

$$\int_M \rho^{1-2\sigma} \frac{|\nabla_g u|^2}{(1+|u|)^\theta} dv_g \leq \frac{1}{\theta-1} \|f\|_{L^1(\partial M)}.$$

*Proof.* In our proofs of this and the next lemma, we adapt some arguments from [6] and [18]. For  $\theta > 0$ , let  $\phi_\theta(r) = \int_0^r \frac{dt}{(1+t)^\theta}$  if  $r \geq 0$  and  $\phi_\theta(r) = -\phi_\theta(-r)$  if  $r < 0$ . It is easy to see that  $\varphi_\theta := \phi_\theta(u) \in H^1(\rho^{1-2\sigma}, M)$  and  $|\varphi_\theta| \leq 1/(\theta-1)$  on  $\overline{M}$  if  $\theta > 1$ . Hence, the Lemma follows from multiplying (60) by letting  $\varphi = \varphi_\theta$ .  $\square$

**Lemma A.5.** Let  $f \in L^2(\partial M)$  with zero mean value,  $u \in \tilde{H}^1$  be the weak solution of (59). Then there exists  $\varepsilon_0 > 0$  depending only on  $n$  and  $\sigma$  such that for any  $1 \leq \tau \leq 1 + \varepsilon_0$ , we have

$$\|u\|_{W^{1,\tau}(\rho^{1-2\sigma}, M)} \leq C,$$

where  $C > 0$  depends only on  $M, g, \sigma, \rho, \|f\|_{L^1(\partial M)}$ .

*Proof.* By the Hölder inequality,

$$\begin{aligned} & \int_M \rho^{1-2\sigma} |\nabla_g u|^\tau dv_g \\ & \leq \left( \int_M \rho^{1-2\sigma} \frac{|\nabla_g u|^2}{(1+|u|)^\theta} dv_g \right)^{\tau/2} \left( \int_M \rho^{1-2\sigma} (1+|u|)^{\frac{\tau\theta}{2-\tau}} dv_g \right)^{(2-\tau)/2} \\ & \leq C(\theta) \left( \int_M \rho^{1-2\sigma} (1+|u|)^{\frac{\tau\theta}{2-\tau}} dv_g \right)^{(2-\tau)/2}, \end{aligned} \quad (61)$$

where we used Lemma A.4 in the last inequality and  $\theta \in (1, 2)$  will be chosen later. Applying Corollary A.2 (see also [17]) to  $\varphi_{\theta/2}$  yields that for any  $1 \leq k \leq 1 + \delta_0$

$$\left( \int_M \rho^{1-2\sigma} |\varphi_{\theta/2} - \fint_M \rho^{1-2\sigma} \varphi_{\theta/2} dv_g|^{2k} dv_g \right)^{1/k} \leq C \int_M \rho^{1-2\sigma} \frac{|\nabla_g u|^2}{(1+|u|)^\theta} dv_g, \quad (62)$$

where  $\delta_0 > 0$  depends only on  $n, \sigma$ , and  $C$  depends only on  $M, g, \sigma, \rho, k$ . Since  $\phi_{\theta/2}(r) \approx |r|^{1-\frac{\theta}{2}}$  for  $|r|$  large, it follows from (62) and Lemma A.4 that

$$\left( \int_M \rho^{1-2\sigma} |u|^{k(2-\theta)} \right)^{1/2k} dv_g \leq C + C \int_M \rho^{1-2\sigma} |u|^{1-\frac{\theta}{2}} dv_g. \quad (63)$$

Choosing  $\theta$  close to 1 such that  $k(2 - \theta) = \frac{\tau\theta}{2-\tau}$  (this can be achieved as long as  $\tau$  is closed to 1) and inserting (63) to (61), we obtain

$$\begin{aligned} \left( \int_M \rho^{1-2\sigma} |\nabla_g u|^\tau dv_g \right)^{1/\tau} &\leq C \left( 1 + \int_M \rho^{1-2\sigma} |u|^{1-\frac{\theta}{2}} dv_g \right)^{\frac{\theta}{2-\theta}} \\ &\leq C + C \left( \int_M \rho^{1-2\sigma} |u| dv_g \right)^{\frac{\theta}{2}} \end{aligned} \quad (64)$$

Since  $\int_M \rho^{1-2\sigma} u dv_g = 0$ , by the Poincaré-Sobolev inequality, Hölder inequality and (64), we have

$$\|u\|_{L^1(\rho^{1-2\sigma}, M)} \leq C \int_M \rho^{1-2\sigma} |\nabla_g u| dv_g \leq C(1 + \|u\|_{L^1(\rho^{1-2\sigma}, M)}^{\frac{\theta}{2}}).$$

Thus,  $\|u\|_{L^1(\rho^{1-2\sigma}, M)} \leq C$  because  $\frac{\theta}{2} < 1$ . Therefore, the lemma follows immediately from (64) and the Poincaré-Sobolev inequality.  $\square$

**Theorem A.4.** *For any bounded radon measure  $f$  defined on  $\partial M$  with  $\langle f, 1 \rangle = 0$ , there exists a weak solution  $u \in W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M)$  of (59).*

*Proof.* The proof follows from Lemma A.3 and A.5 and some standard approximating procedure, see, e.g., [18]. We omit the details here.  $\square$

**Theorem A.5.** *For  $x_0 \in \partial M$ , let  $f = \delta_{x_0} - \frac{1}{|\partial M|_g}$ , where  $|\partial M|_g$  is the area of  $\partial M$  with respect to the induced metric  $g$ . Then there exists a weak solution  $u \in W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M)$  of (59) with mean value zero and for all  $x \in \overline{M} \setminus \{x_0\}$ ,*

$$A_1 \text{dist}_g(x, x_0)^{2\sigma-n} - A_0 \leq u(x) \leq A_2 \text{dist}_g(x, x_0)^{2\sigma-n}, \quad (65a)$$

$$|\nabla_{\tan} u| \leq A_3 \text{dist}_g(x, x_0)^{2\sigma-n-1}, \quad (65b)$$

$$\left| \frac{\partial u}{\partial \nu} \right| \leq A_4 \rho^{2\sigma-1} \text{dist}_g(x, x_0)^{-n}, \quad (65c)$$

where  $A_0, A_1, A_2, A_3, A_4$  are positive constants depending only on  $M, g, n, \sigma, \rho$ .

*Proof.* Let  $f_k \in C^1(\partial M)$  with  $\int_{\partial M} f_k ds_g = 0$ ,  $\|f_k\|_{L^1(\partial M)} \leq C$  independent of  $k$ , such that  $f_k \rightarrow f$  in distribution sense as  $k \rightarrow \infty$ . We can also assume that  $f_k \rightarrow f$  in  $C_{loc}^1(\partial M \setminus \{x_0\})$ . By Lemma A.3 and Lemma A.5, there exists a unique solution  $u_k \in \tilde{H}^1$  of (59) with  $f$  replaced by  $f_k$ , and

$$\|u_k\|_{W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M)} \leq C(\|f_k\|_{L^1(\partial M)}) \leq C.$$

Moreover, it follows from Moser's iterations (see, e.g., the proof of Theorem A.2) that there exists some  $\alpha > 0$  such that

$$\|u_k\|_{C^\alpha(M \setminus \mathcal{B}_r(x_0))} \leq C(r) \quad (66)$$

for any  $r > 0$ . By standard compactness arguments,  $u_k \rightharpoonup u$  in  $W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M)$  for some  $u$ , which is a weak solution of (59) and satisfies

$$\|u\|_{C^{\alpha/2}(M \setminus \mathcal{B}_r(x_0))} \leq C(r).$$

Now, it suffices to establish the estimate (65a) for  $x \in B_r(x_0)$ . For  $r$  suitably small, choose a Fermi coordinate system  $\{y_1, \dots, y_{n+1}\}$  centered at  $x_0$ . Then  $u_k(y)$  satisfies

$$\begin{cases} \partial_i(\rho^{1-2\sigma} \sqrt{\det g} g^{ij} \partial_j u_k) = 0, & \text{in } \mathcal{B}_{2r}^+, \\ -\lim_{y_{n+1} \rightarrow 0} \rho^{1-2\sigma} \sqrt{\det g} \frac{\partial u_k}{\partial y_{n+1}} = f_k, & \text{on } \partial' \mathcal{B}_{2r}^+. \end{cases}$$

Let  $v_k$  be the unique weak solution of

$$\begin{cases} \partial_i(\rho^{1-2\sigma} \sqrt{\det g} g^{ij} \partial_j v_k) = 0, & \text{in } \mathcal{B}_{2r}^+, \\ -\lim_{y_{n+1} \rightarrow 0} \rho^{1-2\sigma} \sqrt{\det g} \frac{\partial v_k}{\partial y_{n+1}} = -\frac{1}{|\partial M|}, & \text{on } \partial' \mathcal{B}_{2r}^+, \\ v_k = u_k & \text{on } \partial'' \mathcal{B}_{2r}^+. \end{cases}$$

in  $H^1(\rho^{1-2\sigma}, M)$ . In view of (66),  $\|v_k\|_{L^\infty(\mathcal{B}_{2r})} \leq C(r)$  and hence  $\|v_k\|_{C^\alpha(\mathcal{B}_r^+)} \leq C(r)$ . Moreover,  $w_k := u_k - v_k \in H^1(\rho^{1-2\sigma}, M)$  satisfies

$$\begin{cases} \partial_i(\rho^{1-2\sigma} \sqrt{\det g} g^{ij} \partial_j w_k) = 0, & \text{in } \mathcal{B}_{2r}^+, \\ -\lim_{y_{n+1} \rightarrow 0} \rho^{1-2\sigma} \sqrt{\det g} \frac{\partial w_k}{\partial y_{n+1}} = f_k + \frac{1}{|\partial M|}, & \text{on } \partial' \mathcal{B}_{2r}^+, \\ w_k = 0 & \text{on } \partial'' \mathcal{B}_{2r}^+. \end{cases}$$

Recall that  $g^{i,n+1} = 0$  for  $i < n+1$  on  $\partial' \mathcal{B}_{2r}^+$ . Let  $\bar{w}_k$  be the even extension of  $w_k$  in  $\mathcal{B}_{2r}$ , i.e.,

$$\bar{w}_k = \begin{cases} w_k(y', y_{n+1}), & y_{n+1} \geq 0, \\ w_k(y', -y_{n+1}), & y_{n+1} \leq 0. \end{cases}$$

We also evenly extend  $g$  and  $\rho$  to be  $\bar{g}$  and  $\bar{\rho}$ , respectively. It is easy to verify that the weak limit  $w$  of  $\bar{w}_k$  in  $L^{1+\varepsilon_0}(\rho^{1-2\sigma}, \mathcal{B}_{2r})$  is the *weak solution vanishing on  $\partial \mathcal{B}_{2r}$*  (see page 162 of [16]) of

$$\partial_i(\bar{\rho}^{1-2\sigma} \sqrt{\det \bar{g}} \bar{g}^{ij} \partial_j w) = -2\delta_0 \quad \text{in } \mathcal{B}_{2r}.$$

It follows from Theorem 3.3 of [16] that  $w$  satisfies the estimates (65a) in  $\mathcal{B}_r(x_0)$ . Thus,  $u$  satisfies (65a). Finally, (65b) and (65c) follows from (65a), Theorem A.3, Proposition A.1 and some scaling arguments.  $\square$

## References

- [1] Adimurthi; Yadava, S. L.: *Some remarks on Sobolev type inequalities*, Calc. Var. Partial Differential Equations **2** (1994), 427–442.
- [2] Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. **11** (1976), no. 4, 573–598.
- [3] Aubin, T.: *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xviii+395 pp. ISBN: 3-540-60752-8.
- [4] Aubin, T.; Li, Y.Y.: *On the best Sobolev inequality*, J. Math. Pures Appl., **78** (1999), 353–387.
- [5] Beckner, W.: *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2) **138** (1993), 213–242.
- [6] Boccardo, L.; Gallouët, T.: *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), 149–169.
- [7] Brezis, H.; Strauss, W. A.: *Semi-linear second-order elliptic equations in  $L^1$* , J. Math. Soc. Japan **25** (1973), 565–590.
- [8] Cabre, X.; Sire, Y.: *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, arXiv:1012.0867.
- [9] Caffarelli, L.; Silvestre, L.: *An extension problem related to the fractional Laplacian*, Comm. Partial. Diff. Equ., **32** (2007), 1245–1260.
- [10] Chang, S.-Y.; González, M.: *Fractional Laplacian in conformal geometry*, Adv. Math. **226** (2011), 1410–1432.
- [11] Druet, O.: *The best constants problem in Sobolev inequalities*, Math. Ann., **314** (1999), 327–346.
- [12] Druet, O.: *Isoperimetric inequalities on compact manifolds*, Geom. Dedicata, **90** (2002), 217–236.
- [13] Druet, O.; Hebey, E.: *The AB program in geometric analysis: sharp Sobolev inequalities and related problems*, Mem. Amer. Math. Soc. **160** (2002), no. 761.
- [14] Escobar, J. F.: *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988), 687–698.
- [15] —: *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. (2) **136** (1992), no. 1, 1–50.
- [16] Fabes, E.; Jerison, D.; Kenig, C.: *The Wiener test for degenerate elliptic equations*, Annales de l’institut Fourier, **32** (1982), 151–182.
- [17] Fabes, E.; Kenig, C. and Serapioni, R.: *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77–116.
- [18] Gallouët, T.; Sire, Y.: *Some possibly degenerate elliptic problems with measure data and non linearity on the boundary*, arXiv:1002.4982v1.
- [19] González, M.: *Gamma convergence of an energy functional related to the fractional Laplacian*, Calc. Var. Partial Differential Equations **36** (2009), 173–210.
- [20] González, M.; Mazzeo, R.; Sire, Y.: *Singular Solutions of Fractional Order Conformal Laplacians*, to appear in J. Geom. Anal..
- [21] González, M.; Qing, J.: *Fractional conformal Laplacians and fractional Yamabe problems*, preprint, arXiv:1012.0579v1.
- [22] Graham, C.R.; Zworski, M.: *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89–118.
- [23] Gurka, P.; Opic, B.: *Continuous and compact imbeddings of weighted Sobolev spaces II*, Czechoslovak Math. J. **39** (1989), 78–94.

- [24] Hebey, E.: “Nonlinear analysis on manifolds: Sobolev spaces and inequalities,” Courant Lecture Notes in Math., 5. New York University, Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI, 1999.
- [25] Hebey, E.; Vaugon, M.: *Meilleures constantes dans le théorème d’inclusion de Sobolev*, Ann. Inst. H. Poincaré, **13** (1996), 57–93.
- [26] Jin, T.; Li, Y.Y.; Xiong, J.: *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, arXiv:1111.1332v1.
- [27] Jin, T.; Li, Y.Y.; Xiong, J.: *On a fractional Nirenberg problem, part II: existence of solutions*, in preparation, 2011.
- [28] Jin, T.; Xiong, J.: *A fractional Yamabe flow and some applications*, arXiv:1110.5664v1.
- [29] Kenig, C.; Pipher, J.: *The Neumann problem for elliptic equations with non-smooth coefficients*, Invent. Math. **113** (1993), 447–509.
- [30] Kufner, A.: *Weighted sobolev spaces*, John Wiley & Sons, Inc., New York, 1985.
- [31] Li, Y.Y.; Ricciardi, T.: *A sharp Sobolev inequality on Riemannian manifolds*, Comm. Pure Appl. Anal. **2** (2003), 1–31.
- [32] Li, Y.Y.; Zhu, M.: *Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries*, Comm. Pure Appl. Math. **50** (1997), 449–487.
- [33] Li, Y.Y.; Zhu, M.: *Sharp Sobolev inequalities involving boundary terms*, Geom. Funct. Anal. **8** (1998), 59–87.
- [34] Lieb, E.H.: *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), 349–374.
- [35] Lions, P.-L.: *The concentration-compactness principle in the calculus of variations, The limit case, II*, Rev. Mat. Iberoamericana **1** (1985), no.2, 45–121.
- [36] Maz’ya, V.: *Sobolev spaces with applications to elliptic partial differential equations*. Second, revised and augmented edition. Springer, Heidelberg, 2011.
- [37] Mazzeo, R.: *The Hodge cohomology of a conformally compact metric*, J. Differential Geom. **28** (1988), 309–339.
- [38] Mazzeo, R.; Melrose, R.: *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260–310.
- [39] Nekvinda, A.: *Characterization of traces of the weighted Sobolev space  $W^{1,p}(\Omega, d_M^\varepsilon)$  on  $M$* , Czechoslovak Math. J., **43** (1993), 695–711.
- [40] Palatucci, G.; Sire, Y.:  *$\Gamma$ -convergence of some super quadratic functionals with singular weights*, Math. Z. **266** (2010), 533–560.
- [41] Qing, J.; Raske, D.: *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds*, Int. Math. Res. Not. 2006, Art. ID 94172, 20 pp.
- [42] Schoen, R.; Yau, S. T.: *Lectures on differential Geometry*. International Press, Cambridge, MA, 1994.
- [43] Tan, J.; Xiong, J.: *A Harnack inequality for fractional Laplace equations with lower order terms*, DCDS-A **31** (2011), 975–983.

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